Mondrian Forests:
Efficient Online Random Forests

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Outline

Background and Motivation

Mondrian Forests
- Mondrian process distribution over $\mathcal{T}$
- Online learning

Experiments

Conclusion

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- **Input**: attributes $X = \{x_i\}_{i=1}^N$, labels $Y = \{y_i\}_{i=1}^N$ (i.i.d)
- $y_i \in \{1, \ldots, K\}$ (classification) or $y_i \in \mathbb{R}$ (regression)
- **Goal**: Predict y_* for test data x_*

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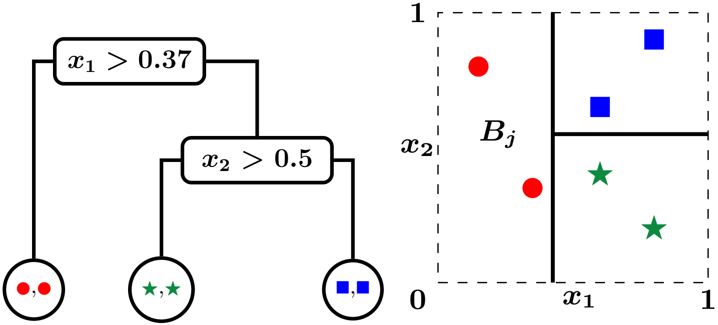
- **Goal**: Predict y_* for test data x_*
- **Recipe for prediction**: Use a 'random forest'
  - Ensemble of randomized decision trees
  - State-of-the-art for lots of real world prediction tasks [Breiman, 2001, Caruana and Niculescu-Mizil, 2006]
  - 'Decision Forests: A Unified Framework for Classification, Regression, Density Estimation, Manifold Learning and Semi-Supervised Learning' [Criminisi et al., 2012]

Example: Classification tree

- Hierarchical axis-aligned binary partitioning of input space
- Rule for predicting label within each block



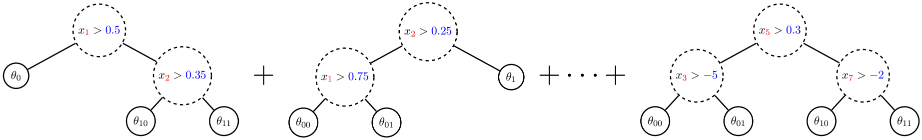
$\mathcal{T}$: list of nodes, feature-id + location of splits for non-leaf nodes
θ: Multinomial parameters at leaf nodes

Random forest (RF)

- Averaged over iid randomized decision trees $\mathcal{T}_1, \ldots, \mathcal{T}_M$ conditioned on X and Y.

$$p(y_*|x_*) = \frac{1}{M} \sum_m p(y_*|x_*, \mathcal{T}_m, X, Y)$$

- Combining multiple decision trees significantly improves predictive performance over single trees.
- Technique for variance reduction, not bias reduction.
- Model combination, not Bayesian model averaging.



Random forest (RF)

- **Breiman's Random Forest** [Breiman, 2001]: Bagging + Randomly subsample features and choose best split amongst subsampled features, optimising over all split locations.
- **Extremely Randomized Trees** [Geurts et al., 2006] (ERT-k): Randomly sample k (feature-id, location) pairs and choose the best split amongst this subset
  - no bagging
  - ERT-1 does not use labels Y to guide splits!

Pros and Cons

- Advantages of RF
  - Excellent predictive performance (test accuracy)
  - Fast to train (in batch setting) and test
  - Trees can be trained in parallel
  - No overfitting

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Mondrian forests = Mondrian process + Random forests

- Can operate in either batch mode or online mode
- Online speed $\mathcal{O}(N \log N)$
- Data efficient (predictive performance of online mode equals that of batch mode!)

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Mondrian process

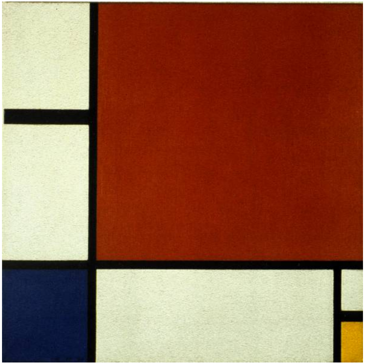
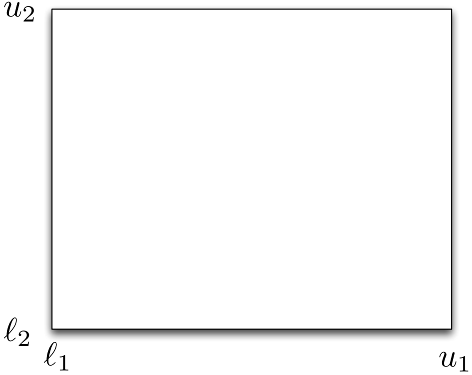


Figure: Mondrian Composition II in Red, Blue and Yellow (Source: Wikipedia)

- A stochastic process over binary hierarchical axis-aligned partitions of $\mathbb{R}^d$ [Roy and Teh, 2009].

Generative process: $\mathcal{MP}(\lambda, [\ell_1, u_1], [\ell_2, u_2])$

1. Draw Δ_ϵ from exponential with rate $u_1 - \ell_1 + u_2 - \ell_2$
2. **IF** $\Delta_\epsilon > \lambda$ stop,



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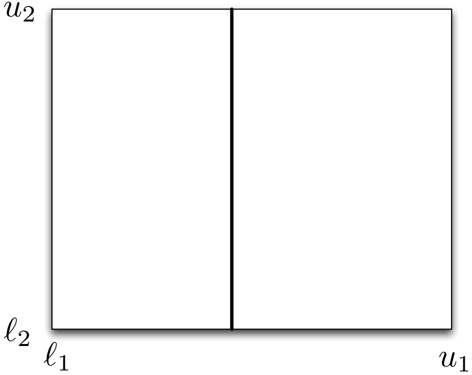
   - Split location: choose cut location uniformly from $[\ell_j, u_j]$



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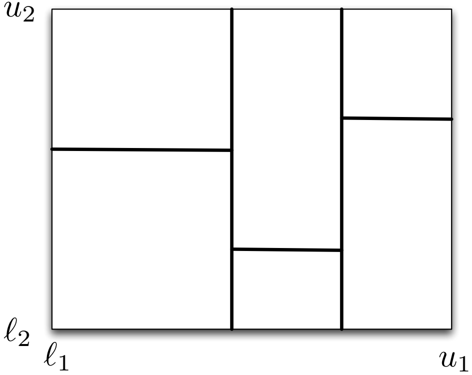
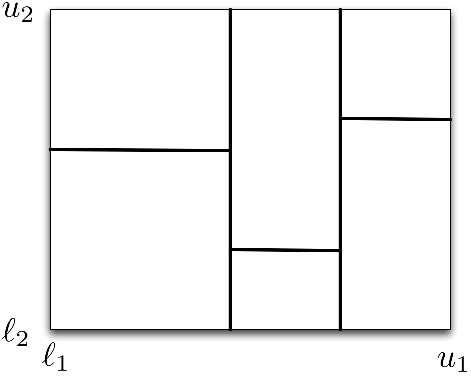
   - Split location: choose cut location uniformly from $[\ell_j, u_j]$
   - Recurse on left and right subtrees with parameter $\lambda - \Delta_\epsilon$



Self-consistency of Mondrian process

- Simulate $\mathcal{T} \sim \mathcal{MP}(\lambda, [\ell_1, u_1], [\ell_2, u_2])$



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- Simulate $\mathcal{T} \sim \mathcal{MP}(\lambda, [\ell_1, u_1], [\ell_2, u_2])$
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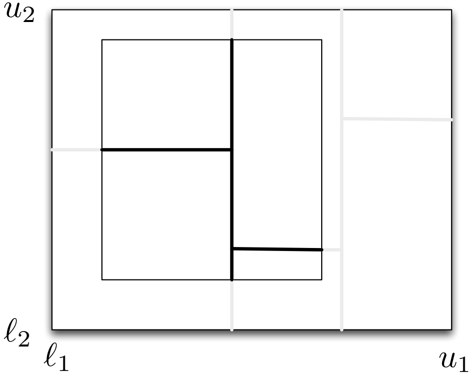

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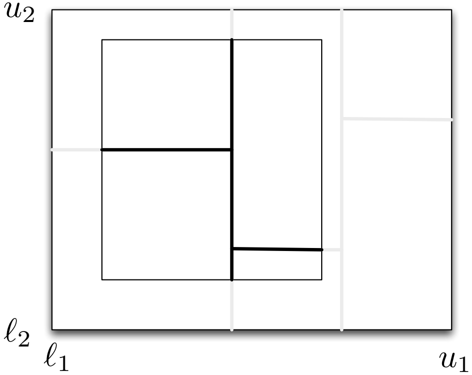

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Self-consistency of Mondrian process

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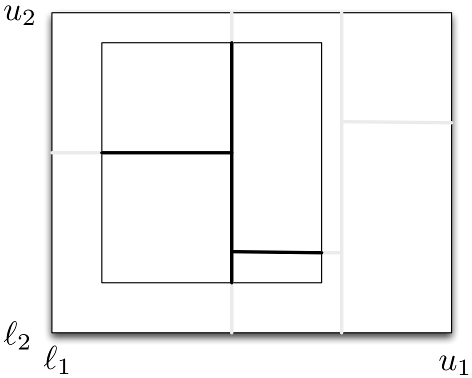

- Restriction has distribution $\mathcal{MP}(\lambda, [\ell_1', u_1'], [\ell_2', u_2'])$!
- Well-defined extension to $\mathcal{MP}(\lambda, \mathbb{R}, \mathbb{R})$, such that $\mathcal{MP}(\lambda, [\ell_1, u_1], [\ell_2, u_2])$ is the restriction to $[\ell_1, u_1] \times [\ell_2, u_2]$.

Mondrian trees

- Use $\mathcal{MP}(\lambda, [\ell_1, u_1], \ldots, [\ell_d, u_d])$ as prior over decision trees $p(\mathcal{T}|X)$, where the range is given by X.

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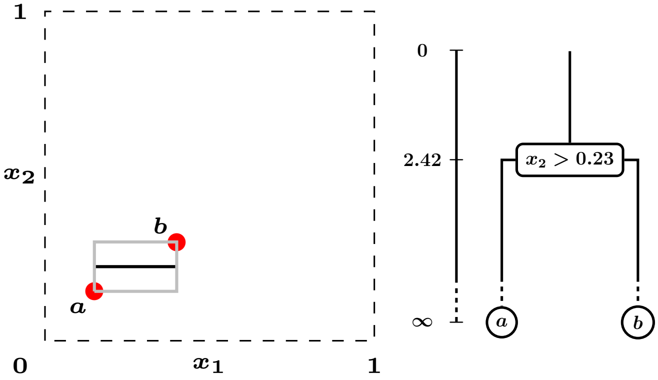
- Online learning:
  - As dataset grows, we simply unveil $\mathcal{T}$ on a larger range.
  - We can enlarge the visible range by simulating from a **conditional Mondrian process**.
  - Distribution of trees in offline and online modes are the same!
  - Order of the data points does not matter.

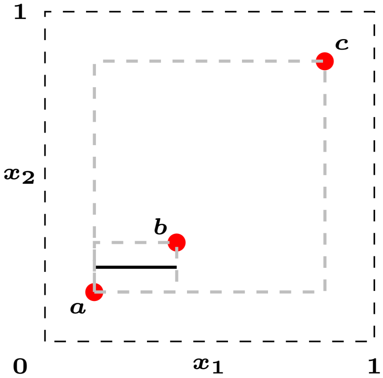
Online learning cartoon

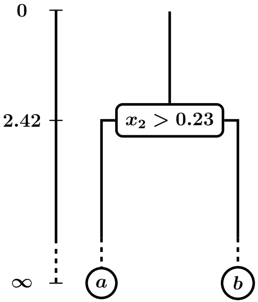
Start with data points *a* and *b*



Online learning cartoon

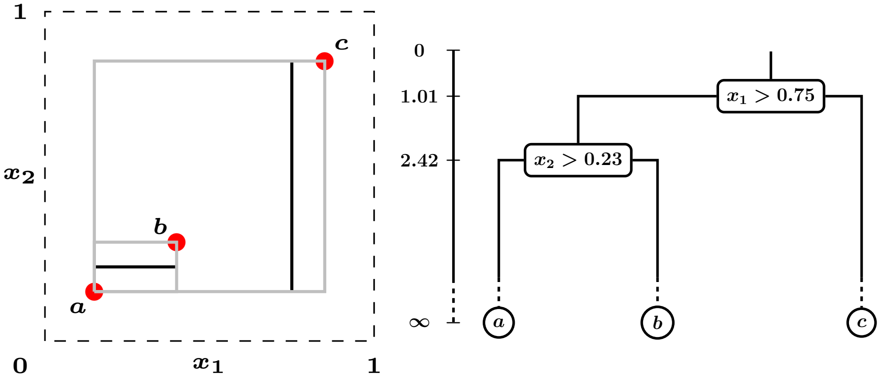
Adding new data point *c*: update range





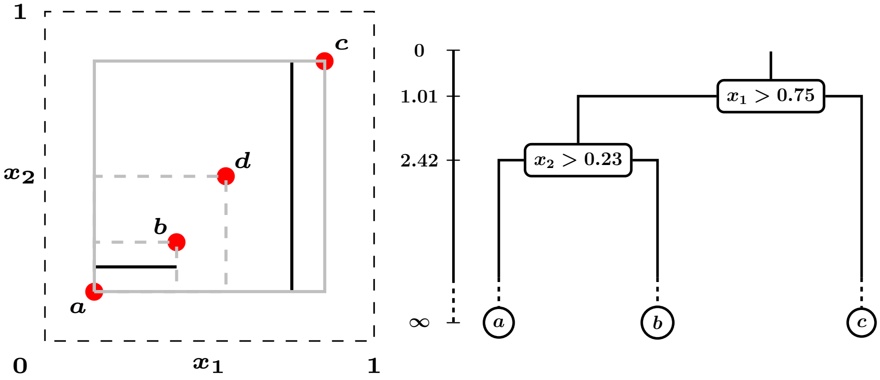
Online learning cartoon

Adding new data point c: introduce new split above existing one



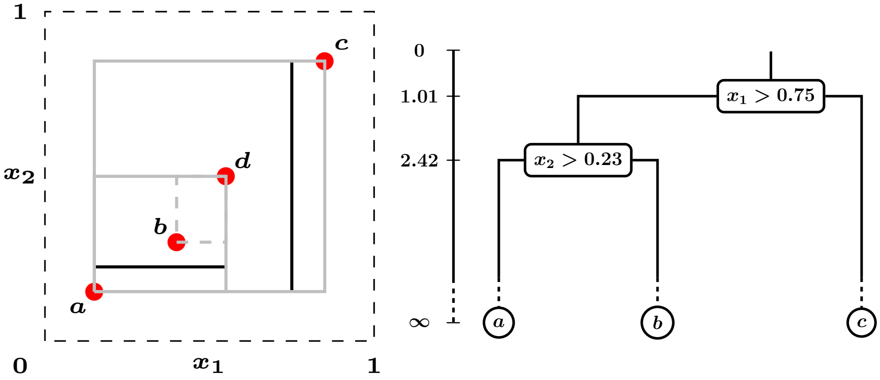
Online learning cartoon

Adding new data point *d*: traverse to left child and update range



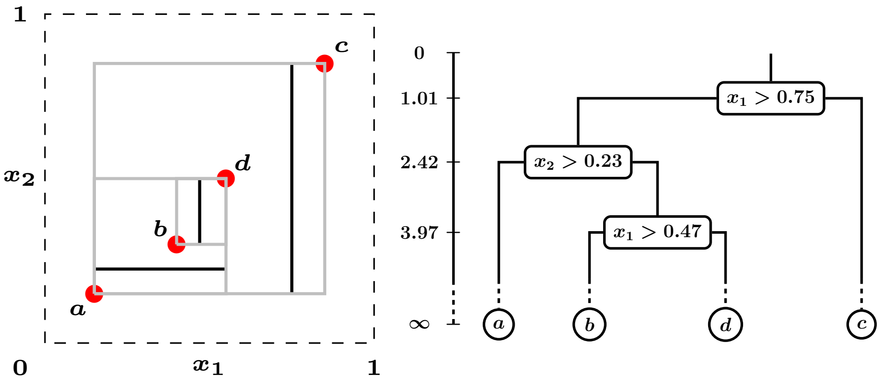
Online learning cartoon

Adding new data point d: extend the existing split to new range



Online learning cartoon

Adding new data point *d*: split leaf further



Key differences between Mondrian forests and existing online random forests

- Splits not extended to unseen regions
- New split can be introduced *anywhere* in the tree (as long as it is consistent with current tree)
- The size and lifetime of a node control probability of new splits being introduced
- Self-consistent hierarchical Bayesian prior on the leaf parameters (not discussed).

Outline

Experimental setup

- Datasets:

Name	D	#Classes	#Train	#Test
Satellite images	36	6	3104	2000
Letter	16	26	15000	5000
USPS	256	10	7291	2007
DNA	180	3	1400	1186

- Training data split into 100 mini batches (unfair to MF)
- Number of trees $= 100$
- Existing randomised decision trees:
  - Periodically retrained offline methods RF, ERT-1, ERT-k.
  - Online RF [Saffari et al., 2009]

Letter

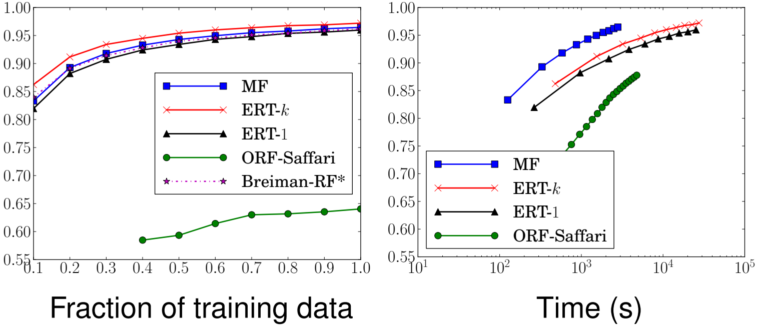


Figure: Test accuracy

- Data efficiency: Online MF very close to offline Breiman's RF and ERT, and significantly outperforms ORF-Saffari.
- Speed: MF much faster than periodically re-trained offline RF and ERT, as well as online RF.

USPS

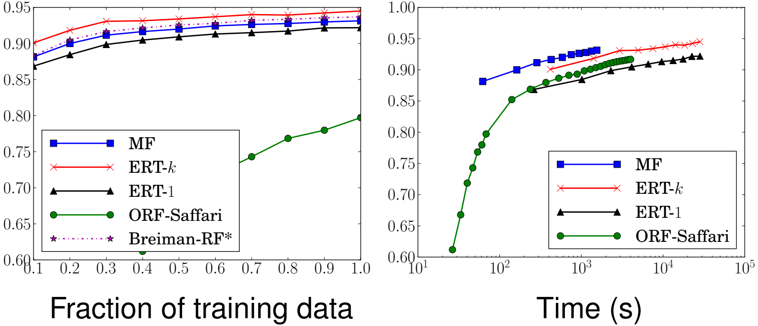
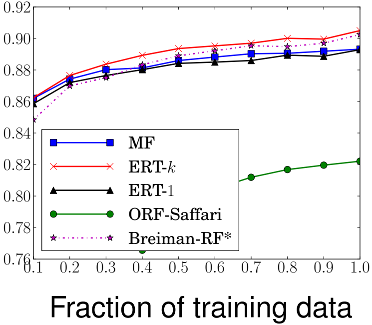
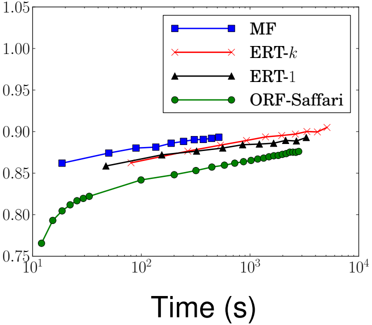


Figure: Test accuracy

Satellite Images





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DNA

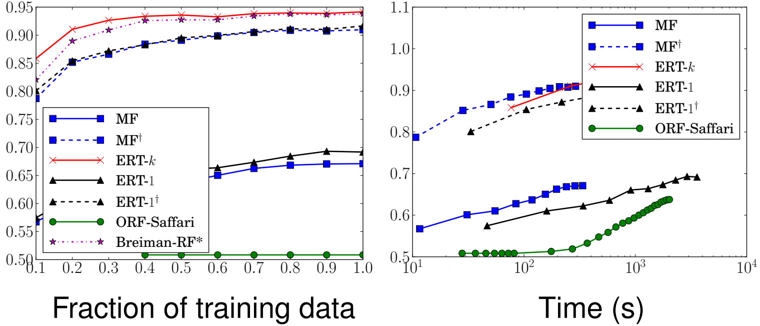

Figure: Test accuracy

- Irrelevant features: Choosing splits independent of labels (MF, ERT-1) harmful in presence of irrelevant features
- Removing irrelevant features (use only the 60 most relevant features[1]) improves test accuracy (MF†, ERT-1†)

[1] https://www.sgi.com/tech/mlc/db/DNA.names

Outline

Conclusion

- MF: Alternative to RF that supports incremental learning
- Computationally faster compared to existing online RF and periodically re-trained Breiman-RF, ERT
- Future work:
  - Mondrian forests for high dimensional data with lots of irrelevant features.
  - Use labels to guide splits in MF (e.g. using ERT-*k* ideas)

Thank you!

arXiv: http://arxiv.org/abs/1406.2673

code: http://www.gatsby.ucl.ac.uk/~balaji/mondrianforest/

Questions?

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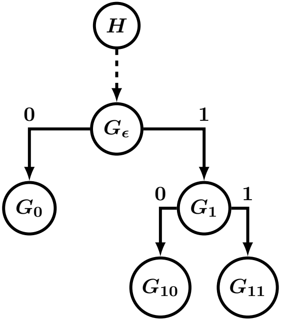
Hierarchical prior over θ

- G_j parametrizes $p(y|x)$ in B_j^x
- Normalized stable process (NSP): special case of PYP where concentration = 0
- $d_j \in (0, 1)$ is discount for node j
- $G_\epsilon | H \sim NSP(d_\epsilon, H)$,
  $G_{j0} | G_j \sim NSP(d_{j0}, G_j)$,
  $G_{j1} | G_j \sim NSP(d_{j1}, G_j)$



- $\mathbb{E}[G_\epsilon(s)] = H(s)$
- $\mathrm{Var}[G_\epsilon(s)] = (1 - d_H)H(s)\big(1 - H(s)\big)$
- Closed under Marginalization: $G_0 | H \sim NSP(d_\epsilon d_0, H)$
- $d_j = e^{-\gamma \Delta_j}$ where Δ_j is the lifetime of node j

Posterior inference for NSP

- Special case of approximate inference for PYP [Teh, 2006]
- Chinese restaurant process representation
- **Interpolated Kneser-Ney smoothing**
  - fast approximation
  - Restrict number of tables serving a dish to at most 1
  - IKN popular smoothing technique in language modeling

Prediction

- Extend Mondrian to range of test data (similar to training)
  - Test data point can potentially branch off and form separate leaf node of its own (unlike conventional decision trees)
  - If test point is in its own node, prediction is made from the (hierarchical) prior
  - Points far away from range of training data are more likely to lie in their own ode
  - We analytically average over every possible extension (unlike training where we sample an extension)
  - Computational complexity linear in tree depth $\approx \log(N)$
- Prediction interpolates between observed labels and prior depending on how close test data point is to training data