A.1. Validation of Observation 1

Assuming the experimental settings described in Section V.A, we consider the approximation of (14), i.e. the relative impact of the term \( \Delta M \Delta v \) in the synthesis error \( \Delta x \). The approximation of (14) involves only one lifting step, hence the experimental assessment is separately carried out for the predict and update step as follows. When \( M = P \), the vector \( x \) comprises the samples of the input signal and the coefficient vector \( v = x^p \) holds the predict-step output. Conversely, when \( M = U \), the input vector is \( x = x^p \) and the coefficient vector is given by \( v = x^u \). The objective is to assess the relative impact of neglecting the term \( \Delta M \Delta v \) in the expression of \( \Delta x \) given by (13), hence we compute the ratio \( \frac{\| \Delta M \Delta v \|}{\| \Delta x \|} \) for several signals \( x \). Prior to synthesis, several perturbation patterns \( \Delta M \) are generated (each with a given mismatch probability \( \rho \)) and quantization is applied to the coefficient vector \( v \) (thereby inducing noise \( \Delta v \)). This leads to a population of synthesis errors \( \Delta x \). Sample results are given in Figure A1 for several probabilities of mismatch \( \rho \in [0.02, 0.14] \). The graphs in the figure report both the average value of the relative approximation error \( \frac{\| \Delta M \Delta v \|}{\| \Delta x \|} \), using dots, and the standard deviation, using bars. As shown in the figure, the approximation of (14) incurs less than a 10% error on average.

---

Figure A1. Relative error incurred by the approximation of (14) with \( M = P \) (left) and \( M = U \) (right) for the cases of medium and fine quantization of the transform coefficients. Several noise matrices \( \Delta M \in \{ \Delta P, \Delta U \} \) representative of different mismatch probabilities are considered: dots denote the average error whereas bars indicate standard deviation.
A.2. Validation of Observation 2

Assuming the experimental settings described in Section V.A, we carried out two complementary set of experiments, measured the average synthesis distortion $E\left\{\|\Delta x\|^2\right\}/T$ and computed the relative approximation error incurred by the expression of (15), both when $M=P$ and $M=U$.

- In the first set of experiments, whose results are reported in Table A1, we consider noise signals $\Delta v$ resulting from increasingly coarse quantization of the transform coefficients (this is indicated in Table A1 by the increasing values of $Q$, which represents the width of the scalar quantizer). Concerning the mismatches in the lifting parameters, which result in the noise matrix $\Delta M$, we consider random mismatches occurring independently for any $t \in \{0, \ldots, T/2 - 1\}$ with probability $\rho = \Pr\left(a[t] \neq \hat{a}[t]\right)$. In essence, this scenario fits the case of independent sources $\Delta M$ and $\Delta v$. Therefore the expression of (15) should represent the observed experimental data with good accuracy. This is confirmed by the results of Table A1, which demonstrate that the relative error incurred by the expression of (15) is 7% on average.

- The second set of experiments, whose results are reported in Table A2 below, investigates the effect of correlation among the mismatches in the lifting parameter and the quantization noise in the transform coefficients due to packet losses. During lifting analysis, we impose that the selection of the lifting filters is kept constant during four consecutive predict-and-update operations. Hence, one adaptive parameter tracks the adaptive decomposition of eight consecutive samples of the input signal. We form individual packets containing both the adaptive parameter and the quantized transform coefficients relative to each segment. During synthesis, the unavailability of one such packet implies that:
  - Four consecutive (and identical) mismatches occur in the adaptive parameter vector and in the resulting noise matrix $\Delta M$.
  - The corresponding transform coefficients are approximated by the coarsest available representation (i.e. the DC component of the corresponding polyphase component).

The results of Table A2 show that relative error incurred by the expression of (15) is 9.5% on average. This suggest that, although each lost packet induces noise samples that are placed at highly correlated locations within $\Delta M$ and $\Delta v$, the values taken by the samples of the two noise sources are independent of each other. As a result, the weak statistical correlation between $\Delta M$ and $\Delta v$ does not induce a major deviation of the values predicted by (15) with respect to the observed data.
A.3. Validation of Observation 3

Assuming the experimental settings described in Section V.B, we generate random mismatched spatial displacements \( \hat{d} \neq d \), which occur within each frame with a certain probability. We then measure the relative error incurred by the approximation (33) for a wide range of quantization accuracies. Sample results obtained using the Football sequence and 5% and 10% mismatch probability are shown in Figure A2. Each curve represents one instantiation of this experiment. As shown in the figure, the approximation (33) incurs errors in the order of 10% in the worst case. We notice that, independently of the mismatched displacement \( \hat{d} \), the approximation of (33) is more accurate when fine-scale quantization is applied. On the other hand, as the overall noise increases due to increasingly coarse quantization, the approximation of (33) progressively overestimates the overall noise power. This behavior agrees with the intuition that the distortion induced by displacement mismatches is “masked” by high quantization noise. In other words, the effect of pointing to the incorrect spatial location within a certain area of the frame is less evident when that area is coarsely quantized.

![Figure A2](image-url)

Figure A2. Relative error incurred by the approximation of (33) for a range of quantization accuracies. Displacement mismatches occur with 5% (left) or 10% probability (right). Different markers denote experiments with different mismatches \( \hat{d} \neq d \).

A.4. Derivation of Observation 4

The following formal definitions are used in subsequent derivations of the expressions (35) and (36).
First, we denote as $D_B (d^p [s, t])$ the block-based multi-reference prediction distortion, i.e. the mean squared error ensuing when predicting a block of $B$ samples, comprising $X [s, 2t + 1]$, using the prediction filter $p_{w [s, t]}$ and the displacement vector $d^p [s, t] = \cdots d^p [s, t] d^p [s, t] d^p [s, t] \cdots$:

$$
D_B (d^p [s, t]) = \frac{1}{B} \sum_{j \in \mathcal{B}(s)} \left( X [s', 2t + 1] + \sum_{j \in \mathcal{J}'} p_{w [s, t]} [B + j] \cdot X [s' - d^p [s, t], 2t + 1 + j] \right)^2
$$

(A1)

where $\mathcal{B}(s)$ denotes the block, comprising the sample $s$, which is treated as a whole during motion-adaptive prediction, and $\mathcal{J}' = \mathcal{J} - \{0\} = \{\pm 1, \pm 3, \ldots, \pm B\}$.

Similarly to the above, we denote as $E_{2t+1} (d^p_j [s, t])$ the sample-wise single-reference prediction error, i.e. the error resulting when the sample $X [s, 2t + 1]$ is predicted from the sample $X [s - d^p_j [s, t], 2t + 1 + j]$:

$$
E_{2t+1} (d^p_j [s, t]) = X [s, 2t + 1] - X [s - d^p_j [s, t], 2t + 1 + j].
$$

(A2)

For each element of the noise vector $\Delta^\text{disp} x^p_{\text{even}} [s + \hat{d}^p]$ in (35), i.e. each time instant $2t$, we consider the worst-case scenario of $B + 1$ mismatched displacements and average all the ensuing errors, thus obtaining:

$$
\Delta^\text{disp} x^p_{\text{even}} [s + \hat{d}^p, 2t] = \frac{1}{B + 1} \sum_{j \in \mathcal{J}'} \left\{ X^p [s - d^p_j [s, t], 2t] - X^p [s - d^p_j [s, t], 2t] \right\}
$$

(A3)

where $t_j = (2t - j - 1)/2$, with $j \in \mathcal{J}'$ as for (A1), and where the sign of the decoding-side displacements is reversed to fit the encoding-side coordinate system. We pursue an approximation of the expected predict-step synthesis distortion (induced by displacement mismatches) which exploits the data already gathered during predict-step analysis. We proceed as follows:

a) First we express the term $\|\Delta^\text{disp} x^p_{\text{even}} [s + \hat{d}^p]\|$ as a function of the differences in the source samples between motion-compensated neighbouring input frames, e.g. $X [s, 2t + 1]$ and $X [s - d^p, 2t]$. The resulting expression aims to involve the prediction residuals (corresponding to odd time instants) that are typical of motion-adaptive temporal prediction.

b) Then we approximate the distortion contribution of individual samples with the average contribution of a block of samples. This links with practical motion-adaptive prediction algorithms that consider blocks rather than individual pixels.

c) Finally we derive an approximation of the term $E \left\{ \|\Delta^\text{disp} x^p_{\text{even}} [s + \hat{d}^p]\| \right\}$ that incorporates a block-based sensitivity term (which reflects the local source characteristics) and the mismatch probability term (which reflects the transmission settings).

(a) Sample-wise distortion induced by displacement mismatches in predict-step synthesis: Under the assumption that prediction errors relative to different instants (i.e. resulting when samples within different input frames are predicted) are temporally orthogonal, the term $\|\Delta^\text{disp} x^p_{\text{even}} [s + \hat{d}^p]\|$ can be expressed as:

$$
\|\Delta^\text{disp} x^p_{\text{even}} [s + \hat{d}^p]\|^2 = \sum_{t=0}^{T/2-1} \sum_{j \in \mathcal{J}'} \left( \frac{E_{2t+1} (d^p_j [s, t])}{B + 1} \right)^2 + \left( \frac{E_{2t+1} (d^p_j [s, t])}{B + 1} \right)^2 - \left( \frac{2E_{2t+1} (d^p_j [s, t]) E_{2t+1} (d^p_j [s, t])}{(B + 1)^2} \right).
$$

(A4)
The expression (A4) is obtained by first adding and subtracting the term \( X[s,2t_j+1] \) inside the summation of (A3) yielding:
\[
\Delta_{\text{disp},x_{\text{even}}}^{s}[s + \hat{d}^p,2t] = \frac{1}{B} + \sum_{j \in \mathcal{J},j'} \left[ \mathcal{E}_{2t+1}(d^p[s,t]) - \mathcal{E}_{2t+1}(\hat{d}^p[s,t]) \right]
\]
(A5)
where the error \( \mathcal{E}_{2t+1}(d^p[s,t]) \) ensues as the sample \( X[s,2t_j+1] \) is predicted from the sample \( X[s-d^p[s,t]],2t_j+j+1 \). Similarly for \( \mathcal{E}_{2t+1}(\hat{d}^p[s,t]) \). The hypothesis that prediction errors relative to different instants are temporally orthogonal implies that \( \sum_{i=0}^{T/2-1} [\mathcal{E}_{2t+1}(d^p[s,t]) \cdot \mathcal{E}_{2t+1}(d^p[s,t])] = 0 \) when \( i \neq l \), irrespectively of both \( d^p[s,t] \) and \( \hat{d}^p[s,t] \). Evaluating \( \|\Delta_{\text{disp},x_{\text{even}}}^{s}[s + \hat{d}^p]\| \) using (A5) then leads to (A4).

(b) Approximation to block-wise distortion: The sample-wise error terms in (A4) are approximated with the block-based equivalent formulation as follows. First we approximate the distortion relative to single pixels with the distortion contribution of small areas of the frames. In other words:
\[
\sum_{j \in \mathcal{J},j'} \left[ \frac{1}{B} + \sum_{j \in \mathcal{J},j'} \left[ p_{s'_j|s} [B+j] \cdot \mathcal{E}_{2t+1}(d^p[s,t]) \right]\right]^2 
\]
(A6)
Assuming that prediction errors from different reference frames (e.g. \( \mathcal{E}_{2t+1}(d^p[s,t]) \) and \( \mathcal{E}_{2t+1}(d^p[s,t]) \), \( j \neq l \)) are spatially orthogonal and recalling that \( \sum_{j \in \mathcal{J},j'} p_{s'_j|s} [B+j] = -1 \), the approximation (A6) becomes:
\[
\sum_{j \in \mathcal{J},j'} \left[ \frac{1}{B} + \mathcal{E}_{2t+1}(d^p[s,t]) \right]^2 \approx \sum_{j \in \mathcal{J},j'} D_B(d^p[s,t])
\]
(A7)
Using (A7), we approximate equation (A4) as:
\[
\|\Delta_{\text{disp},x_{\text{even}}}^{s}[s + \hat{d}^p]\| \approx \sum_{i=0}^{T/2-1} \left[ D_B\left(\hat{d}^p[s,t]\right) - D_B\left(d^p[s,t]\right) \right] + \sum_{i=0}^{T/2-1} \mathcal{X}_{2t+1}^B\left(d^p[s,t],\hat{d}^p[s,t]\right)
\]
(A8)
with:
\[
\mathcal{X}_{2t+1}^B\left(d^p[s,t],\hat{d}^p[s,t]\right) = \frac{2}{B} \sum_{j \in \mathcal{J},j'} \sum_{j \in \mathcal{J},j'} \left[ \mathcal{E}_{2t+1}(d^p[s',t])^2 \right] \left[ \frac{\mathcal{E}_{2t+1}(d^p[s',t])}{B+1} \right] \left[ 1 - \frac{\mathcal{E}_{2t+1}(\hat{d}^p[s',t])}{\mathcal{E}_{2t+1}(d^p[s',t])} \right]
\]
(A9)
Neglecting the contribution of the terms \( \mathcal{X}_{2t+1}^B \) in the above incurs less than 15% error in practice and allows simplifying the approximation (A8) as:
\[
\|\Delta_{\text{disp},x_{\text{even}}}^{s}[s + \hat{d}^p]\| \approx \sum_{i=0}^{T/2-1} \left[ D_B\left(\hat{d}^p[s,t]\right) - D_B\left(d^p[s,t]\right) \right]
\]
(A10)
During the motion estimation phase, several suitable candidate displacements \( \hat{d}^p \) are tested and the corresponding block-based distortions \( D_{2t+1}^B\left(\hat{d}^p[s,t]\right) \) are measured. The displacement yielding the minimum distortion (the “correct” value \( d^p \)) is then selected to perform analysis. Such values \( D_{2t+1}^B\left(d^p[s,t]\right) \) and \( D_{2t+1}^B\left(\hat{d}^p[s,t]\right) \) are used in (A10).

(c) Expected distortion induced by displacement mismatches in predict-step synthesis: The expression (35), which approximates the distortion induced by displacement mismatches \( E\left\{ \left\|\Delta_{\text{disp},x_{\text{even}}}^{s}[s + \hat{d}^p]\right\| \right\} \), is derived by taking statistical expectation of (A10) over the probability that \( \hat{d}^p \) is used, thus obtaining:

\[ JUNE 22, 2010 \]
\[
E \left\{ \| \Delta \text{dist} \mathbf{x}^p [s + \mathbf{d}] \|^2 \right\} \approx \sum_{t=0}^{T/2-1} \sum_{\mathbf{d} \in \mathcal{D}_B \left( \mathbf{d}^\circ \right)} \mathbf{D}_B \left( \mathbf{d}^\circ \right) \cdot \Pr \left( \mathbf{d}^\circ [s, t] = \mathbf{d}^\circ [s, t] \right) \cdot \Pr \left( \mathbf{d}^\circ [s, t] = \mathbf{d}^\circ [s, t] \right)
\]

\[
(A11)
\]

where \( \Pr \left( \mathbf{d}^\circ [s, t] = \mathbf{d}^\circ [s, t] \right) \) denotes the probability that a displacement mismatch occurs and \( \Pr \left( \mathbf{d}^\circ [s, t] = \mathbf{d}^\circ [s, t] \right) \) denotes the probability that the displacement \( \mathbf{d}^\circ [s, t] \) is used in case of mismatch. Assume that, in case of a mismatch, any candidate displacement (as tested during motion estimation) can be used to perform synthesis. Let \( \mathcal{N} \left( \mathbf{d}^\circ [s, t] \neq \mathbf{d}^\circ [s, t] \right) \) denote the number of such displacements. Therefore \( \Pr \left( \mathbf{d}^\circ [s, t] = \mathbf{d}^\circ [s, t] \right) = 1/\mathcal{N} \left( \mathbf{d}^\circ [s, t] = \mathbf{d}^\circ [s, t] \right) \) and (A11) becomes (35).

### APPENDIX B

#### B.1. Proofs of Proposition 1 and Corollary 1

**Proof of Proposition 1:** Using the SVD [28] of the matrix \((2\mathbf{I}-\mathbf{M})\) to express \( \left\langle (2\mathbf{I}-\mathbf{M}) \mathbf{\Delta v} \right\rangle \) yields:

\[
\frac{1}{T} E \left\{ \| \left(2\mathbf{I}-\mathbf{M} \right) \mathbf{\Delta v} \|^2 \right\} = \frac{1}{T} \sum_{i=1}^{T} \mathbf{c}_i \mathbf{q}_i \left(2\mathbf{I}-\mathbf{M}\right) \mathbf{c}_i \mathbf{q}_i \left(2\mathbf{I}-\mathbf{M}\right) \mathbf{c}_i \mathbf{q}_i \left(2\mathbf{I}-\mathbf{M}\right)
\]

\[
(B1)
\]

with \( \mathbf{c}_i \{2\mathbf{I}-\mathbf{M}\} \) and \( \mathbf{q}_i \{2\mathbf{I}-\mathbf{M}\} \) as given by Definition 1. We then recall that:

\[
(b^T c)^2 = \text{tr} \left\{ (cc^T) (bb^T) \right\} = \text{tr} \left\{ (bb^T) (cc^T) \right\}
\]

\[
(B2)
\]

where \( b \) and \( c \) are \( T \times 1 \) vectors. Using (B2) in (B1), interchanging the trace and expectation operators, and combining the linearity of the trace operator with (1) leads to (16).

**Proof of Corollary 1:** By expanding (16) we have:

\[
\frac{1}{T} E \left\{ \| \left(2\mathbf{I}-\mathbf{M} \right) \mathbf{\Delta v} \|^2 \right\} = \frac{1}{T} \sum_{k=0}^{T} \sum_{j=0}^{T} \left( \sum_{k=0}^{T} \sum_{j=0}^{T} W^{(2\mathbf{I}-\mathbf{M})} [k, k] R_{\Delta v} [k, k] + \frac{2}{T} \sum_{j=0}^{T} \sum_{k=0}^{T} W^{(2\mathbf{I}-\mathbf{M})} [k, k-j] R_{\Delta v} [k, k-j] \right). \]

\[
(B3)
\]

The hypothesis made for \( \Delta v_{\text{even}} \) and \( \Delta v_{\text{odd}} \) implies that the second term on the right hand side of (B3) is zero. Furthermore, we have \( R_{\Delta v} [2k, 2k] = E \left\{ \| \Delta v_{\text{even}} \|^2 \right\} / (T/2) \) and \( R_{\Delta v} [2k+1, 2k+1] = E \left\{ \| \Delta v_{\text{odd}} \|^2 \right\} / (T/2) \) for \( k = 0, 1, \ldots, T/2 \). Therefore separating the even and odd values of \( k \) in (B3) yields (17).

#### B.2. Proof of Proposition 2

**Lemma 1:** Let \( \eta \in \{1,2,\ldots,T/2\} \) denote the number of synthesis lifting parameters that do not match their analysis counterpart, i.e. \( \hat{a}[t] = a[t] \) at \( \eta \) distinct time instants. The induced synthesis distortion is:

\[
\frac{1}{T} E \left\{ \| \Delta \mathbf{M} \mathbf{v} \|^2 \mid \eta \right\} = \frac{1}{T} \text{tr} \left\{ \mathbf{v} \mathbf{W} \mathbf{W} \mathbf{v} \right\}
\]

\[
(B4)
\]

where the \( T \times T \) matrix \( \mathbf{W} \{ \Delta \mathbf{M}_\eta \} \) is defined in (21).

**Proof:** The distortion induced by a given matrix \( \Delta \mathbf{M} \in \Delta \mathbf{M}_\eta \) and coefficient vector \( \mathbf{v} \) is given by (16) and (B2) as \( \| \Delta \mathbf{M} \mathbf{v} \|^2 / T = \left\{ \frac{1}{T} \text{tr} \left\{ \mathbf{v} \mathbf{v}^T \right\} \right\} \mathbf{W} \{ \Delta \mathbf{M} \} \). Performing statistical average over each \( \Delta \mathbf{M} \in \Delta \mathbf{M}_\eta \), recalling the definition (21) and exploiting linearity, yields the expression (B4) for \( E \left\{ \| \Delta \mathbf{M} \mathbf{v} \|^2 \mid \eta \right\} / T \).
Proof of Proposition 2: \( E \left\{ \| \Delta M v \|^2 \right\} / T = \left( 1 / T \right) \sum_{\eta=1}^{T/2} \left[ \text{Pr} \left( \eta \right) E \left\{ \| \Delta M v \|^2 \mid \eta \right\} \right] \). Since \( \Delta M_0 = \{0\} \) then \( E \left\{ \| \Delta M v \|^2 \mid \eta = 0 \right\} = 0 \). Employing (B4) for \( \eta > 0 \) yields (20).

B.3. Proof of Proposition 3:

Lemma 2: Assuming that \( \Delta x_{\text{even}}^n \), \( \Delta x_{\text{odd}}^n \) and the coefficients on the even rows of \( \Delta U \) constitute three mutually independent white WSS noise processes, we have:

\[
E \left\{ \Delta x^p [2i+1] \Delta x^p [2j] \right\} = \begin{cases} E \left\{ \| \Delta x_{\text{odd}}^p \|^2 \right\} / \left( T / 2 \right) u_{(i,j)} \left[ L^p + \left( 2i+1 \right) j \right] & , \left[ 2i-2j+1 \right] \leq L^p \\ 0 \quad , \left[ 2i-2j+1 \right] > L^p \end{cases} \quad \text{(B5)}
\]

\[
E \left\{ \Delta x^p [2i] \Delta x^p [2j+1] \right\} = \begin{cases} E \left\{ \| \Delta x_{\text{odd}}^p \|^2 \right\} / \left( T / 2 \right) u_{(i,j)} \left[ L^p + \left( 2i+2 \right) j \right] & , \left[ 2j-2i+1 \right] \leq L^p \\ 0 \quad , \left[ 2j-2i+1 \right] > L^p \end{cases} \quad \text{(B6)}
\]

where \( u_{n} [k] \) is as in (3). Furthermore, for \( i \neq j \), we have:

\[
E \left\{ \Delta x^p [2i] \Delta x^p [2j] \right\} = \begin{cases} E \left\{ \| \Delta x_{\text{odd}}^p \|^2 \right\} / \left( T / 2 \right) \left[ \sum_{h=0} \ u_{(0,j)} \left[ L^p + h \right] u_{(i,j)} \left[ L^p + \left( 2i+2 \right) j + h \right] \right] & , \left| i-j \right| \leq L^p \\ 0 \quad , \left| i-j \right| > L^p \end{cases} \quad \text{(B7)}
\]

where \( i' = \{ \pm 1, \pm 3, \ldots, \pm L^p \} \).

Proof: It follows via simple algebraic derivation recalling that \( \Delta x^p = (2I-U) \Delta x^p - \Delta U x^p - \Delta U x^p \).

Proof of Proposition 3: From (15) we have \( E \left\{ \| \Delta x^p \|^2 \right\} / T = E \left\{ \| \left( 2I-P \right) \Delta x^p \|^2 \right\} / T + E \left\{ \| \Delta P x^p \|^2 \right\} / T \).

Expressing the first term using (B3) (since \( \Delta x^n \) is not white) and the second term using (20) yields:

\[
\frac{1}{T} E \left\{ \| \Delta x^p \|^2 \right\} = \frac{1}{T} \sum_{\eta=1}^{T/2} \left[ \text{Pr} \left( \eta \right) \text{tr} \left( \left( x^p \left( x^p \right)^T \right) W \left\{ \Delta P_\eta \right\} \right) \right] + \gamma_c \left\{ P \right\} E \left\{ \| \Delta x_{\text{even}}^p \|^2 \right\} / \left( T / 2 \right) + \gamma_c \left\{ P \right\} E \left\{ \| \Delta x_{\text{odd}}^p \|^2 \right\} / \left( T / 2 \right) \quad \text{(B9)}
\]

Since \( E \left\{ \| \Delta x_{\text{odd}}^p \|^2 \right\} / \left( T / 2 \right) = E \left\{ \| \Delta x_{\text{even}}^p \|^2 \right\} / \left( T / 2 \right) \) we derive \( E \left\{ \| \Delta x_{\text{even}}^p \|^2 \right\} / \left( T / 2 \right) \) from \( E \left\{ \| \Delta x^p \|^2 \right\} / T \), which is in turn obtained applying (15) (17) and (20) to the update step. The last term in (B9) equals

\[
\frac{2}{T} \sum_{j=2,4, \ldots}^{T/2} \sum_{k,j+2, \ldots}^{T/2} W^{\left( 2L \right)} \left( k, k-j \right) E \left\{ \Delta x^p \left[ k \right] \Delta x^p \left[ k-j \right] \right\} + \frac{2}{T} \sum_{j=1,3, \ldots}^{T/2} \sum_{k,j+1,3, \ldots}^{T/2} W^{\left( 2L \right)} \left( k, k-j \right) E \left\{ \Delta x^p \left[ k \right] \Delta x^p \left[ k-j \right] \right\} \quad \text{(B10)}
\]

Using (B5)-(B8) the expression (B10) becomes \( \left[ \xi \left\{ P, U \right\} E \left\{ \| \Delta x_{\text{odd}}^p \|^2 \right\} / \left( T / 2 \right) \right] \), with\(^7\):

\[
\xi \left\{ P, U \right\} = \alpha \left\{ P, U \right\} - \beta \left\{ P, U \right\} \quad \text{(B11)}
\]

\(^7\)The odd samples of \( \left( 2I-U \right) \Delta x^n \) and \( \Delta x^n \) coincide whereas the odd samples of \( \Delta U x^n \) are zero, hence \( \Delta x_{\text{odd}}^p = \Delta x_{\text{odd}}^n \).

\(^8\)The approximated expressions of \( \alpha \left\{ P, U \right\} \) and \( \beta \left\{ P, U \right\} \), isolating the contributions of \( P \) and \( U \), are used in practice.
\[ \alpha \{ P, U \} = \frac{2}{T} \sum_{j=2,4,...}^{2L^2} \left[ \sum_{k,j+2,...}^{T-2} W^{(2L^2)}(k,k-j) \right] \sum_{k \in Z} \left( u^{(k+j,2)} u^{(k+j-1)} [L^p+h] u^{(k+j-2)} [L^p+j+h] \right) \]

\[ \beta \{ P, U \} = \frac{2}{T} \sum_{j=2,4,...}^{2L^2} \left[ \sum_{k,j+2,...}^{T-2} W^{(2L^2)}(k,k-j) \right] \left[ \frac{2}{TL^2} \sum_{l=0}^{T/2-1} \sum_{k \in Z} \left( u^{(k+l)} [L^p+h] u^{(k+l-j)} [L^p+j+h] \right) \right] \]

Using the short-hand of (23)-(24) the above leads to:

\[ \frac{1}{T} E \{ \| \Delta x \|^2 \} = \varphi_0 \{ P, U \} \frac{E \{ \| \Delta x_{\text{even}} \|^2 \}}{T/2} + \varphi_1 \{ P, U \} \frac{E \{ \| \Delta x_{\text{odd}} \|^2 \}}{T/2} \]

\[ + \frac{1}{T} \sum_{\eta=1}^{T/2} \left\{ \Pr(\eta) \left[ \text{tr} \left\{ \left( x^p \left( x^p \right)^T \right) W \{ \Delta P_\eta \} \right\} \right] + 2\gamma_0 \{ P \} \text{tr} \left\{ \left( x^u \left( x^u \right)^T \right) W \{ \Delta U_\eta \} \right\} \right\} \]

Since errors in the lifting parameters occur independently, \( \Pr(\eta) = \rho^\eta \left( 1 - \rho \right)^{T/2-\eta} \left( \frac{T/2}{\eta} \right) \). Therefore, replacing

\[ \psi \{ \rho, P, x^p, x^u \} = \sum_{\eta=1}^{T/2} \left[ \rho^\eta \left( 1 - \rho \right)^{T/2-\eta} \left( \frac{T/2}{\eta} \right) \right] \left[ \text{tr} \left\{ \left( x^p \left( x^p \right)^T \right) W \{ \Delta P_\eta \} \right\} \right] + 2\gamma_0 \{ P \} \text{tr} \left\{ \left( x^u \left( x^u \right)^T \right) W \{ \Delta U_\eta \} \right\} \]

[where \( W \{ \Delta P_\eta \} \) and \( W \{ \Delta U_\eta \} \) are given by (21)] in (B14) leads to (22).