New Bounds for Error Probability Analysis in Digital Communications

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Abstract: *General bounds for error probability analysis in digital communications are examined with particular attention to effecting tightening of the widely used Chernoff bound. Weighting functions are studied, leading to tighter integral and series forms for upper bounds and identification of the forms available for lower bounds.*

1. Introduction

In digital communication systems design, the theoretical expression representing the error probability P_e is often so complicated that it is impractical to use in error probability analysis. Instead, we often work with a simpler expression, representing an upper bound for P_e [1]. The general inequality most commonly used to obtain such expressions, is called the Chernoff bound [2]. It is simple to apply but is not very tight. This paper establishes new inequalities that tighten the Chernoff bound. Lower bounds, while of secondary importance, are also considered.

2. Formulation

The problem addressed in this paper may be expressed in the following abstract form:

Given that f(x) is a non-negative valued function of a real variable, that F(z) is a complex valued function, defined on the complex numbers by

$$F(z) = \int_{-\infty}^{\infty} f(x) e^{zx} dx \tag{1}$$

and that T has the value

$$T = \int_{A}^{\infty} f(x) dx \tag{2}$$

find bounds for T in terms of F and A, in particular, tighten the Chernoff bound

$$T \le e^{-As} F(s) \quad \textit{for all } s > 0 \tag{3}$$

Firstly, by defining

$$g(x) = f(x + A) \qquad G(z) = e^{-Az} F(z) \tag{4}$$

the problem reduces to the form:

Given that g(x) is a non-negative valued function of a real variable, that G(z) is a complex valued function, defined on the complex numbers by

$$G(z) = \int_{-\infty}^{\infty} g(x) e^{zx} dx \tag{5}$$

and that T has the value

$$T = \int_{-\infty}^{\infty} g(x) U(x) dx \tag{6}$$

where U(x) is the unit step function, find bounds for T in terms of G, in particular, tighten the Chernoff bound

$$T \le G(s) \quad \text{for all } s > 0 \tag{7}$$

3. Weighting Functions

Order relations between bounding functions and T, rely on order relations between weighting functions and U, and the following simple implication, which uses our assumption that g is non-negative valued:

$$W_1(x) \le W_2(x) \Rightarrow \int_{-\infty}^{\infty} g(x) W_1(x) dx \le \int_{-\infty}^{\infty} g(x) W_2(x) dx \tag{8}$$

The following list of weighting functions, defined for $s > 0$, have order relations that allow us to apply (8), and integral or series representations that allow us to express the resulting integrals in terms of G.

(i) $$U(x) = \frac{1}{2} e^{sx} \left(e^{-s|x|} + \operatorname{sgn}(x) e^{-s|x|} \right) = \frac{1}{2\pi} e^{sx} \left(\int_{-\infty}^{\infty} \frac{s \cos ux}{s^2 + u^2} du + \int_{-\infty}^{\infty} \frac{u \sin ux}{s^2 + u^2} du \right) \tag{9}$$

(ii) $$W_{\infty,\infty}(x) = e^{s(x-|x|)} = \frac{1}{\pi} e^{sx} \int_{-\infty}^{\infty} \frac{s \cos ux}{s^2 + u^2} du \tag{10}$$

(iii) $$W_{\infty,L}(x) = e^{sx} H(x) \tag{11}$$

where H is the Fourier series of $e^{-s|x|}$ *on the interval* $[-L, L]$

$$H(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos u_k x \quad \text{with} \quad a_k = \frac{2s\left(1 - (-1)^k e^{-Ls}\right)}{L\left(s^2 + u_k^2\right)},\ u_k = \frac{k\pi}{L} \tag{12}$$

(iv) $$W_{n,L}(x) = e^{sx} H_n(x) \tag{13}$$

where

$$H_n(x) = \left(\frac{a_0}{2} - \sum_{k=n+1}^{\infty} a_k \right) + \sum_{k=1}^{n} a_k \cos u_k x = \left(1 - \sum_{k=1}^{n} a_k \right) + \sum_{k=1}^{n} a_k \cos u_k x \tag{14}$$

(v) $$w_{\infty,\infty}(x) = \operatorname{sgn}(x) e^{s(x-|x|)} = \frac{1}{\pi} e^{sx} \int_{-\infty}^{\infty} \frac{u \sin ux}{s^2 + u^2} du \tag{15}$$

It is easy to show that the above weighting functions satisfy the order relations

$$U(x) \le W_{\infty,\infty}(x) \le W_{\infty,L}(x) \le \cdots \le W_{n,L}(x) \le \cdots \le W_{1,L}(x) \le W_{0,L}(x) = e^{sx} \tag{16}$$

$$U(x) \ge w_{\infty,\infty}(x) \tag{17}$$

4. Bounding Functions

The corresponding exact expression for T [3] and bounding functions are:

(i) $$T = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \frac{s \operatorname{Re} G(s+iu)}{s^2+u^2} du + \int_{-\infty}^{\infty} \frac{u \operatorname{Im} G(s+iu)}{s^2+u^2} du \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(s+iu)}{s+iu} du \tag{18}$$

(ii) $$B_{\infty,\infty}(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s \operatorname{Re}(G(s+iu))}{s^2+u^2} du = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \operatorname{Re}(G(s+is\tan\theta)) d\theta \tag{19}$$

(iii) $$B_{\infty,L}(s) = \frac{a_0}{2} G(s) + \sum_{k=1}^{\infty} a_k \operatorname{Re} G(s+iu_k) \tag{20}$$

(iv) $$B_{n,L}(s) = \left(1 - \sum_{k=1}^{n} a_k\right) G(s) + \sum_{k=1}^{n} a_k \operatorname{Re} G(s+iu_k) \tag{21}$$

(v) $$b_{\infty,\infty}(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u \operatorname{Im}(G(s+iu))}{s^2+u^2} du \tag{22}$$

By (8), (16), and (17), the above bounding functions satisfy the order relations

$$T \le B_{\infty,\infty}(s) \le B_{\infty,L}(s) \le \cdots \le B_{n,L}(s) \le \cdots \le B_{1,L}(s) \le B_{0,L}(s) = G(s) \tag{23}$$

$$T \ge b_{\infty,\infty}(s) \tag{24}$$

The upper bound methods cannot be applied further than our solitary lower bound. This is because, if h is periodic and $h(x) \le U(x)e^{-sx}$, then $h \le 0$. Therefore we cannot improve upon the lower bound zero with a series bound. This forces us to consider weighting functions of the form $w(x) = e^{sx} p(x)$ where $p(x)$ is a polynomial $p(x) = \sum_{k=0}^{n} c_k x^k$. If $U \ge w$, the corresponding bound is

$$T \ge \sum_{k=0}^{n} c_k G^{(k)}(s) \tag{25}$$

These differential bounds look less promising and difficult to obtain. Since lower bounds are less important, they will not be pursued further here, except to note that the difficulty with lower bounds suggests why approximations to T with error terms are more complicated than upper bounds.

5. Optimisation of Bounding Functions

It can be shown that by choosing n, L, and s sufficiently large, we can make $B_{n,L}(s)$ arbitrarily close to T. A question of more practical importance is this: for a given n, how do we chose L and s to minimise $B_{n,L}(s)$?. This is a complicated issue. Clearly, the answer depends upon the specific function G. However, there are some general observations to be made.

(i) If s_0 minimises the Chernoff bound i.e. $G(s_0) = \min\{G(s) : s > 0\}$ then the integrals $\int_{-\infty}^{0} g(x)e^{s_0 x}dx$ and $\int_{0}^{\infty} g(x)e^{s_0 x}dx$ are both small, in which case s_0 is also a reasonable choice for making $B_{n,L}(s)$ small, especially if s_0 is large.

(ii) The following bound provides some information for making a sensible choice of L.

$$\frac{1}{2L}\int_{-L}^{L}\left(H_n(x) - H(x)\right)^2 dx \le \frac{3}{2}\left(1 - \frac{2}{\pi}\arctan\frac{(n+1)\pi}{Ls}\right)^2 \qquad (26)$$

6. Summary

A general method has been developed for finding variants of the Chernoff bound. This has been used with the general aim of maximising the accuracy of the bound while minimising the complexity of the bounding function. Bounding functions in integral form $B_{\infty,\infty}(s)$ and finite series form $B_{n,L}(s)$ have been obtained, tighter than the Chernoff bound. Some general insights have gained about the choice of parameters for $B_{n,L}(s)$. A lower bound in integral form $b_{\infty,\infty}(s)$ has been derived and the forms available for lower bounds have been noted – integrals or differential series.

The utility of the bounds $B_{\infty,\infty}(s)$ and $B_{n,L}(s)$ for error probability analysis in digital communication systems looks to merit further investigation.

References.

[1] K.W. Cattermole, 'Mathematical foundations of communications design vol.2: Statistical analysis and finite structure', Pentech, London, 1986.

[2] J.J. O'Reilly, 'Bounds and approximations in probability theory', chapter 4 in: 'Mathematical topics in telecommunications – vol.2: Problems of randomness in communication theory', Ed. K.W Cattermole and J.J O'Reilly, Pentech, London, 1994.

[3] V.K. Prabhu, 'Modified Chernoff bounds for PAM systems with noise and interference', IEEE Trans. Inform. Theory, vol. IT-28, pp. 95-100, Jan. 1982.