On the Distribution of the Intermodulation Distortion in OFDM Communication Systems

M. R. D. Rodrigues and J. J. O'Reilly
Department of Electronic and Electrical Engineering
University College London
{mrodrigu, j.oreilly}@ee.ucl.ac.uk

Abstract: In this paper the distribution of the intermodulation distortion in OFDM communication systems is studied. In particular, we observe that while the distribution appears to converge to a Gaussian distribution for a high number of OFDM sub-channels, it is definitely not Gaussian for a low number of OFDM sub-channels (approximately 48) as previously thought.

1. Introduction

OFDM is a multi-carrier modulation scheme that has been used for digital audio broadcasting (DAB), digital video broadcasting (DVB) and has recently been proposed for indoor wireless networks (e.g., HIPERLAN2). This recent interest on OFDM has been motivated by its capability to handle frequency selective fading and by its bandwidth efficiency. The major drawback of OFDM is its vulnerability to non-linear distortion.

To evaluate the performance of OFDM signals in non-linear communication systems the distribution of the intermodulation distortion has to be known accurately. In this context, a Gaussian approximation has often been used [1,2,3,4,5,6] but it was recently found that such an approximation is not always appropriate [7].

The purpose of this paper is to study the distribution of the intermodulation distortion in OFDM communication systems. Particularly, we study the distribution of the intermodulation distortion for OFDM/BPSK signals.

2. The Intermodulation Distortion

In this section we show the impact of a non-linearity on a simple OFDM communication system. In figure 1 we show the model of the non-linear OFDM communication system under consideration. In this figure, $s(t)$ is the complex envelope of the transmitted OFDM signal, $r(t)$ is the complex envelope of the received OFDM signal, $n(t)$ is the additive white complex Gaussian noise and the bandpass non-linearity is represented by its equivalent baseband kernels, $h_n(t_1,¼,t_n)$, $n=1,3,¼$, or by its equivalent baseband transfer functions, $H_n(f_1,¼,f_n)$, $n=1,3,¼$ [8]. S_{kn} is the complex transmitted symbol in time slot k and sub-channel n and R_{kn} is the complex received symbol in time slot k and sub-channel n; $g_n(t-kT)$ denotes the complex waveform used to convey the information in time slot k and sub-channel n and $g_n^*(t-kT)$ denotes its complex conjugate. Finally, the integrate and dump filter integrates over $kT+T_{CP}$ and $(k+1)T$ where T is the OFDM symbol duration and T_{CP} is the OFDM cyclic prefix duration. Note that we have chosen to represent the bandpass non-linearity by a bandpass Volterra series so that both frequency independent behaviour and frequency dependent behaviour could be accounted for. Examples of frequency independent and frequency dependent non-linearities can be found in [1,2,3,4,5,6] and [9,10] respectively.

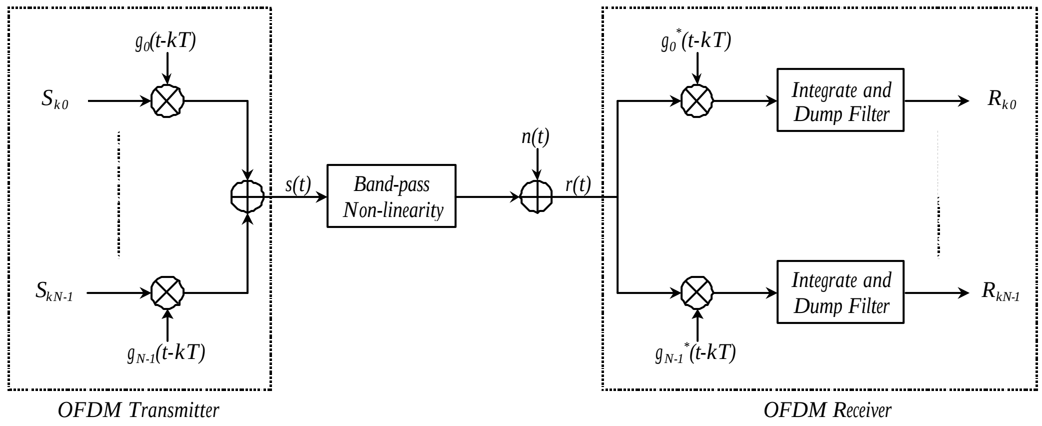


Figure 1: Model of a simple non-linear OFDM communication system.

We now wish to identify the various contributions to the symbol received in a particular sub-channel, specifically the intermodulation distortion. To accomplish this task we first note that the complex envelope of the transmitted OFDM signal is given by

$$s(t)=\sum_{k=-\infty}^{\infty}\sum_{n=0}^{N-1}S_{kn}g_n(t-kT) \tag{1}$$

$$g_n(t)=\begin{cases}\dfrac{1}{\sqrt{T-T_{CP}}}e^{j\frac{2\pi n(t-T_{CP})}{T-T_{CP}}} & t\in[0,T]\\ 0 & t\notin[0,T]\end{cases} \tag{2}$$

and the complex envelope of the received OFDM signal is given by [8]

$$\begin{aligned}r(t)=&\int_{-\infty}^{\infty}h_1(\tau_1)s(t-\tau_1)d\tau_1+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_3(\tau_1,\tau_2,\tau_3)s(t-\tau_1)s(t-\tau_2)s^*(t-\tau_3)d\tau_1d\tau_2d\tau_3+\cdots\\&+n(t)\end{aligned} \tag{3}$$

The symbol received in time slot k and sub-channel n is obtained by multipying the received OFDM signal by the complex conjugate of the complex waveform used to convey the information in that time slot and that sub-channel and then integrating over the useful OFDM symbol duration, i.e., over $kT+T_{CP}$ and $(k+1)T$. Thus,

$$R_{kn}=\int_{kT+T_{CP}}^{(k+1)T}r(t)g_n^*(t-kT)dt \tag{4}$$

If we now assume that $h_1(t_1)$ is non-zero only for $t_1 \in [0,T_{CP}]$; $h_3(t_1,t_2,t_3)$ is non-zero only for $t_1,t_2,t_3 \in [0,T_{CP}]$; etc.[1] then the symbol received in time slot k and sub-channel n is given by

$$\begin{aligned}R_{kn}=&\sum_{n_1=0}^{N-1}S_{kn_1}\frac{1}{(T-T_{CP})^0}H_1\left(\frac{n_1}{T-T_{CP}}\right)\delta(n_1-n)\\&+\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\sum_{n_3=0}^{N-1}S_{kn_1}S_{kn_2}S_{kn_3}^*\frac{1}{(T-T_{CP})^1}H_3\left(\frac{n_1}{T-T_{CP}},\frac{n_2}{T-T_{CP}},-\frac{n_3}{T-T_{CP}}\right)\delta(n_1+n_2-n_3-n)+\cdots\\&+N_{kn}\end{aligned} \tag{5}$$

where

$$N_{kn}=\int_{kT+T_{CP}}^{(k+1)T}n(t)g_n^*(t-kT)dt \tag{6}$$

Note that the various contributions to the symbol received in sub-channel n can be identified on the right hand side of eq. 5. The first set of terms corresponds to the linear/first-order contribution of the non-linearity, the second set of terms corresponds to the cubic/third-order contribution of the non-linearity (the third-order intermodulation distortion), etc. and N_{kn} is a complex Gaussian random variable.

Finally, it is important to observe that we are not generally concerned with the distribution of the intermodulation distortion in a particular sub-channel as given by the second, etc. set of terms on the right hand side of eq. 5 but the distribution of the intermodulation distortion conditioned on a given transmitted symbol in that particular sub-channel (e.g., $-A$ or A in the BPSK case). Moreover, although the intermodulation distortion is a complex valued random variable, in the BPSK case we are not concerned with its two-dimensional distribution but with the one-dimensional distribution of its real part. These details are explained in [7] where an analytic technique was developed to determine the impact of non-linearities on the error probability of OFDM/BPSK signals.

In the next section the distribution of the intermodulation distortion for OFDM/BPSK signals is studied by simulation. Note that in this case the term intermodulation distortion reflects the changes referred to in the last paragraph.

---

[1] For frequency independent non-linearities these conditions are always met because the kernels consist of Dirac delta pulses; for frequency dependent non-linearities these conditions are usually met because the cyclic prefix duration is large.

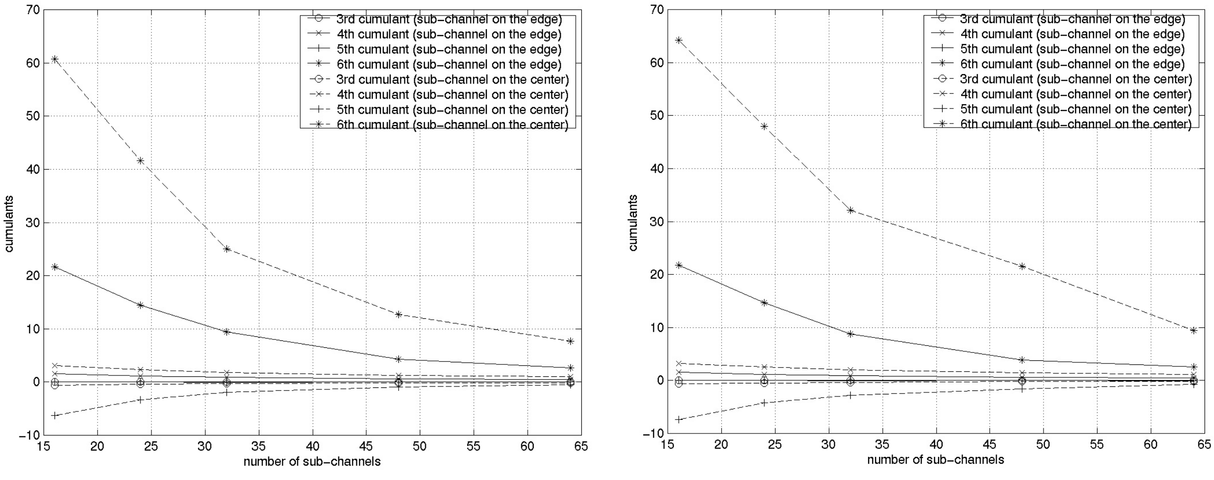
3. Simulation Results

As mentioned before, in this section the distribution of the intermodulation distortion for OFDM/BPSK signals is studied by simulation. We have considered both a frequency independent non-linearity and a frequency dependent non-linearity. The frequency independent non-linearity consists of an instantaneous non-linearity (a third-order polynomial-law device) and a zonal filter in series. The frequency dependent non-linearity consists of an RLC bandpass filter, an instantaneous non-linearity (a third-order polynomial-law device), another RLC bandpass filter and a zonal filter in series. The centre frequency and the 3dB-bandwidth of the RLC bandpass filter equal the centre frequency and the 3dB-bandwidth of the OFDM signal respectively. The purpose of the zonal filter is to eliminate all the harmonics but the first so that the non-linearity is bandpass. This filter does not introduce any distortion in the band of interest. Note that for these two simple non-linearities the kernels or transfer functions of order greater than three are zero [8].

We have studied the distribution of the intermodulation distortion for OFDM/BPSK signals with different number of sub-channels (16-64). In each case we have considered the intermodulation distortion in the sub-channel on the edge of the band, where the number of intermodulation products is lower, and in the sub-channel on the centre of the band, where the number of intermodulation products is higher. We have assumed that the complex transmitted symbols were independent and took any value belonging to a BPSK constellation $\{-A,A\}$ with equal probability. We have also assumed that the complex transmitted symbol in sub-channel n, where n corresponds to the sub-channel being studied, took the value $-A$ (we note that except for a different mean the distribution of the intermodulation distortion given $-A$ or given A is identical due to symmetry).

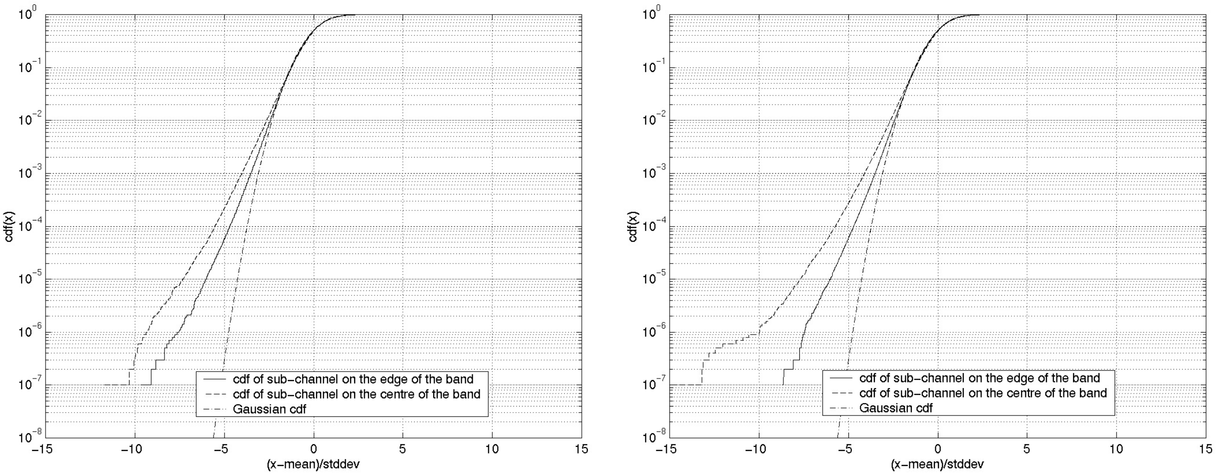
In figure 2 a number of cumulants are shown against the number of OFDM sub-channels for the frequency independent non-linearity and for the frequency dependent non-linearity. Solid lines correspond to the sub-channel on the edge of the band and dashed lines correspond to the sub-channel on the centre of the band. For a Gaussian distribution the first cumulant corresponds to its mean, the second cumulant corresponds to its variance and all the other higher order cumulants are zero. From figure 2 we observe that the higher order cumulants tend to zero and hence the Gaussian approximation will ultimately be valid. We also observe that the Gaussian convergence is faster for the sub-channels on the edge of the band than for the sub-channels on the centre of the band. Finally, we notice that the speed of convergence does not seem to depend much on the nature of the non-linearity, i.e., whether the non-linearity is frequency independent or frequency dependent.

To conclude, in figure 3 we show the cumulative distribution function of the intermodulation distortion for a 48 sub-channel OFDM/BPSK signal for the frequency independent non-linearity and for the frequency dependent non-linearity. Again, solid lines correspond to the sub-channel on the edge of the band and dashed lines correspond to the sub-channel on the centre of the band. From these figures we conclude that for a 48 sub-channel OFDM/BPSK signal a Gaussian approximation is not appropriate. In fact, it was shown in [7] that when computing the error probability of a non-linearly distorted 48 sub-channel OFDM/BPSK signal a Laplace approximation yields better results when compared to a Gaussian approximation.



(a) **(b)**

Figure 2: Cumulants vs. number of OFDM sub-channels. (a) Frequency independent non-linearity; (b) frequency dependent non-linearity.



(a) **(b)**

Figure 3: Cumulative distribution function of the intermodulation distortion for a 48 sub-channel OFDM/BPSK. (a) Frequency independent non-linearity; (b) frequency dependent non-linearity.

4. Conclusions and Future Work

In this paper we have observed that while the distribution of the intermodulation distortion appears to converge to a Gaussian distribution for a high number of OFDM sub-channels, it is definitely not Gaussian for a low number of OFDM sub-channels (approximately 48) as previously thought. Consequently, for OFDM communication systems that use a relatively low number of sub-channels (e.g., HIPERLAN2) some other approximation ought to be used.

Finally, we notice that in this paper we have only considered OFDM/BPSK signals and mild non-linearities, i.e., non-linearities whose kernels or transfer functions of order greater than three are zero. Future work will include OFDM/M-PSK and OFDM/M-QAM signals and strong non-linearities, i.e., non-linearities whose kernels or transfer functions of order greater than three are not zero.

5. Acknowledgments

This work has been supported by *Fundação para a Ciência e a Tecnologia* through the *PRAXIS XXI* scholarship programme.

6. References

[1] D. Dardari, V. Tralli and A. Vaccari, "A theoretical characterization of nonlinear distortion effects in OFDM systems," IEEE Transactions on Communications, vol. COM-48, pp. 1755-1764, October 2000.

[2] P. Banelli and S. Cacopardi, "Theoretical analysis and performance of OFDM signals in nonlinear AWGN channels," IEEE Transactions on Communications, vol. COM-48, pp. 430-441, March 2000.

[3] E. Costa, M. Midrio and S. Pupolin, "Impact of amplifier nonlinearities on OFDM transmission system performance," IEEE Communications Letters, vol. 3, pp. 37-39, February 1999.

[4] G. Santella and F. Mazzenga, "A hybrid analytical-simulation procedure for performance evaluation in M-QAM-OFDM schemes in presence of nonlinear distortions," IEEE Transactions on Vehicular Technology, vol. 47, pp. 142-151, February 1998.

[5] C. L. Liu, "The effect of nonlinearity on a QPSK-OFDM-QAM signal," IEEE Transactions on Consumer Electronics, vol. 43, pp. 443-447, August 1997.

[6] Q. Shi, "OFDM in bandpass nonlinearity," IEEE Transactions on Consumer Electronics, vol. 42, pp. 253-258, August 1996.

[7] M. R. D. Rodrigues, "An analytic technique to assess the impact of non-linearities on the error probability of OFDM signals in RoF based wireless networks," Proceedings of the IEEE ISIT 2001, Washington, D.C., U.S.A., pp. 316, June 2001.

[8] S. Benedetto, E. Biglieri and R. Daffara, "Modelling and performance evaluation of nonlinear satellite links – a Volterra series approach," IEEE Transactions on Aerospace and Electronic Systems, vol. AES-15, pp. 494-506, July 1979.

[9] M. R. D. Rodrigues, "Error probability assessment of radio over fibre based wireless networks employing OFDM signalling," Proceedings of the London Communications Symposium 2000, London, U.K., September 2000.

[10] M. R. D. Rodrigues, "Assessment of the performance of radio over fibre based wireless networks employing OFDM signalling," Proceedings of the London Communications Symposium 1999, London, U.K., July 1999.