# An Investigation of MIMO Single-Carrier Frequency-Domain MMSE Equalization

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**Abstract:** In recent years, the ambition to achieve very high data rates in wireless communication systems has fuelled the study of multiple-input multiple-output (MIMO) architectures and has resulted in the need for more advanced equalization techniques. Currently, the wireless industry is at a point where typical time-domain equalization solutions are becoming too complex to implement in MIMO systems. In this paper, a single-carrier frequency-domain equalization (SC-FDE) solution is presented that utilizes the minimum mean-squared error (MMSE) criterion to perform equalization on the received message. The error-rate performance of a system that employs this MMSE equalizer is analyzed.

## 1 Introduction

As wireless communication systems have evolved over the past few years, the demand for higher data rates has grown enormously. As a direct consequence of the increase in data rates, the effects of multipath propagation and intersymbol interference (ISI) have been augmented, resulting in the need for more complex equalizers. The complexity of the equalization process is enlarged even more in MIMO systems due to the addition of extra transmit and receive antennas. The wireless communication industry is approaching a point in time where the desired bandwidth of a MIMO system will require equalization solutions that are far too complex to be implemented with standard time-domain equalization (TDE) structures. Because of this fact, several alternative reduced-complexity equalization techniques are currently being studied. One such alternative, namely single-carrier frequency-domain equalization (SC-FDE), derives its relatively low complexity from its use of the frequency domain to perform equalization on a block of received symbols. The benefits of SC-FDE have been explored for many single-input single-output (SISO) architectures [1]. In this paper, an SC-FDE technique for use in a MIMO system is presented. This technique employs the MMSE criterion to perform equalization on a received message.

The rest of this paper is organized as follows. In section 2, a brief overview of an SC-FDE system is presented. The derivation for the MIMO MMSE equalizer is given in section 3. The results of computer simulations are illustrated in section 4. Finally, conclusions are given in section 5.

## 2 System Overview

SC-FDE utilizes the frequency domain to equalize blocks of received symbols. In order to understand the concept of frequency-domain equalization, let us first consider time-domain equalization. In a conventional linear wideband wireless communication system, the received signal sampled at symbol intervals is mathematically defined as the linear convolution between the discrete-time transmitted sequence and the discrete-time channel impulse response (CIR). To equalize a received signal of this nature in the time domain with a typical symbol-spaced linear transversal equalizer (LTE), the received sequence is essentially linearly convolved with the equalizing filter's tap coefficients. Generally, as data rates increase the number of resolvable taps in the delay profile of the channel increases, which requires a larger filter to equalize the received signal. As the filter becomes very large (e.g. in the order of 20+ taps), time-domain equalization with a conventional LTE becomes too complex to practically implement and other equalization alternatives must be explored [2]. One such alternative is SC-FDE.

In an uncoded SISO SC-FDE system, equalization is performed on a block of received symbols corresponding to $N$ transmitted data symbols. To perform this equalization, a *cyclic prefix* must be added to each block of transmitted symbols and removed from each corresponding block of received symbols to eliminate *interblock interference* (IBI). In addition to eliminating IBI, the implementation of a cyclic prefix creates the illusion of periodicity in the system, converting the *linear* convolution between the transmitted block and the CIR as performed in a conventional wideband linear system to a *circular* convolution. This conversion allows equalization to be performed by *cyclically* convolving the received

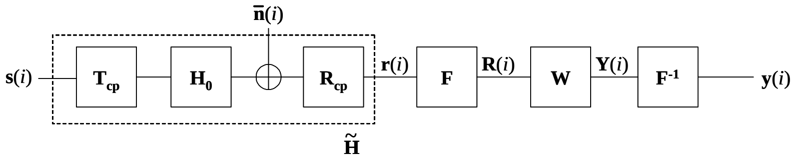


Figure 1: Block diagram of a SISO SC-FDE system.

sequence with the appropriate equalizer coefficients. This method of equalization may still prove to be too complex to implement. However, the periodic nature of the SC-FDE system allows the fast Fourier transform (FFT) and the inverse fast Fourier transform (IFFT) to be employed in order to efficiently equalize each received block [3]. A block diagram of a SISO SC-FDE system is illustrated in Figure 1. In this block diagram, $\mathbf{s}(i)$ is the $i$th transmitted block of $N$ data symbols, $\mathbf{T_{cp}}$ represents the addition of a cyclic prefix to the block $\mathbf{s}(i)$, $\mathbf{H}_0$ depicts the linear channel, $\bar{\mathbf{n}}(i)$ is a Gaussian noise process, $\mathbf{R_{cp}}$ illustrates the cyclic prefix removal step, $\widetilde{\mathbf{H}}$ represents the equivalent circular channel, $\mathbf{F}$ and $\mathbf{F}^{-1}$ are the FFT and IFFT operations, respectively, $\mathbf{W}$ is the frequency-domain equalizer, and $\mathbf{y}(i)$ is the $i$th equalized received block.

In a MIMO system the signals from each of the transmit antennas are conveyed across the channel and summed at each receive antenna. Consequently, a means of separating these signals must be available at the receiver. Solutions to this problem involving the use of space-time codes have be presented [4]. Here, we propose a method by which the received symbol blocks are separated and equalized in the same process by employing the MMSE criterion without employing redundant symbols. The proposed MIMO SC-FDE system may be represented as shown in Figure 2 where $n_T$ is the number of transmit antennas, $n_R$ is the number of receive antennas, and $\widehat{\mathbf{W}}$ is the frequency-domain equalizer. For the rest of the paper, it is assumed that $n_T = n_R = n$.

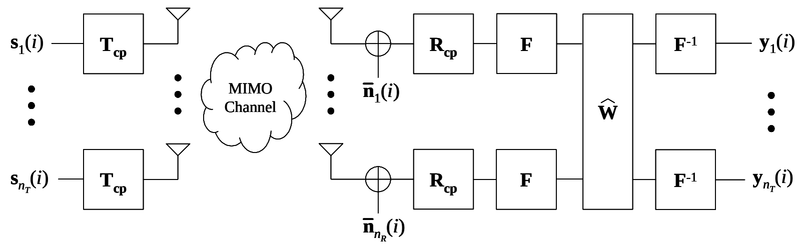


Figure 2: Block diagram of a MIMO SC-FDE system.

## 3 Derivation of MIMO MMSE Equalizer

To derive the MIMO MMSE equalizer, we adopt a matrix representation of the SC-FDE system where each element of the block diagram in Figures 1 and 2 represents a matrix operation. Let the $i$th block of $N$ symbols transmitted from the $p$th transmit antenna be denoted by $\mathbf{s}_p(i) = \{s_{p,1}(i),\, s_{p,2}(i),\, \ldots,\, s_{p,N}(i)\}^T$ where the superscript $^T$ indicates the transpose operation. Now, consider all $n$ blocks of transmitted symbols at time $i$ given by

$$\mathbf{s}(i) = \left\{\mathbf{s}_1(i)^T, \mathbf{s}_2(i)^T,\, \ldots,\, \mathbf{s}_n(i)^T\right\}^T. \tag{1}$$

Observing Figures 1 and 2, it can be shown that the $n$ equalized received blocks corresponding to the $n$ transmitted blocks at time $i$ can be represented by

$$\mathbf{y}(i) = \mathbf{D_F}^{-1}\widehat{\mathbf{W}}\mathbf{D_F}\widetilde{\mathbf{H}}\mathbf{s}(i) + \mathbf{D_F}^{-1}\widehat{\mathbf{W}}\mathbf{D_F}\tilde{\mathbf{n}}(i) \tag{2}$$

where $\mathbf{D_F}$ is an $n$-by-$n$ block-diagonal matrix with each matrix on the diagonal equal to the $N$-by-$N$ FFT matrix $\mathbf{F}$, $\tilde{\mathbf{n}}(i)$ is an $nN$-by-1 vector of complex Gaussian noise terms following the removal of the cyclic prefix, each with zero mean and variance equal to $\sigma_n^2/2$ per dimension, and $\widetilde{\mathbf{H}}$ is the block-channel matrix given by the following equation.

$$\widetilde{\mathbf{H}} = \begin{pmatrix} \widetilde{\mathbf{H}}_{1,1} & \widetilde{\mathbf{H}}_{1,2} & \cdots & \widetilde{\mathbf{H}}_{1,n} \\ \widetilde{\mathbf{H}}_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \widetilde{\mathbf{H}}_{n,1} & \cdots & \cdots & \widetilde{\mathbf{H}}_{n,n} \end{pmatrix}. \tag{3}$$

where $\widetilde{\mathbf{H}}_{p,q}$ is a circulant matrix defining the channel between the $q$th transmit antenna and the $p$th received antenna. Equation (2) suggests that the matrix $\widehat{\mathbf{W}}$ not only performs equalization on the received message but separates the received symbols as well.

The goal is to construct an equalizer matrix $\widehat{\mathbf{W}} = \widehat{\mathbf{W}}_{\mathbf{MMSE}}$ such that the expectation of the squared Euclidean distance between the equalized received signal and the corresponding transmitted signal is minimized. The equalizer matrix $\widehat{\mathbf{W}}_{\mathbf{MMSE}}$ can be defined mathematically by the following equation.

$$\widehat{\mathbf{W}}_{\mathbf{MMSE}} = \underset{\widehat{\mathbf{W}}}{\arg\min}\, \mathrm{E}\left[|\mathbf{s}(i) - \mathbf{y}(i)|^2\right] \tag{4}$$

where $\mathrm{E}[\cdot]$ denotes the expectation operation. Using equation (2) and omitting the block index $i$ for simplicity, the expectation in equation (4) can be expanded to yield

$$\begin{aligned} \mathrm{E}\left[|\mathbf{s} - \mathbf{y}|^2\right] \;=\; & \sigma_s^2 \mathbf{I}_{nN} - \sigma_s^2 \widetilde{\mathbf{H}}^H \mathbf{D_F}^{-1} \widehat{\mathbf{W}}^H \mathbf{D_F} - \sigma_s^2 \mathbf{D_F}^{-1} \widehat{\mathbf{W}} \mathbf{D_F} \widetilde{\mathbf{H}} \\ & + \sigma_s^2 \mathbf{D_F}^{-1} \widehat{\mathbf{W}} \mathbf{D_F} \widetilde{\mathbf{H}} \widetilde{\mathbf{H}}^H \mathbf{D_F}^{-1} \widehat{\mathbf{W}}^H \mathbf{D_F} + \sigma_n^2 \mathbf{D_F}^{-1} \widehat{\mathbf{W}} \widehat{\mathbf{W}}^H \mathbf{D_F} \end{aligned} \tag{5}$$

where $\mathbf{I}_{nN}$ is the $nN$-by-$nN$ identity matrix, $\sigma_s^2$ is the power of the transmitted signal, and the superscript $^H$ indicates the conjugate transpose operation. To find the matrix $\widehat{\mathbf{W}}$ that minimizes equation (5), we must take the partial derivative of equation (5) with respect to $\widehat{\mathbf{W}}^H$, equate the resulting expression to the $nN$-by-$nN$ zero matrix, and solve for $\widehat{\mathbf{W}}$. Following these steps gives us the MMSE equalizer matrix

$$\widehat{\mathbf{W}}_{\mathbf{MMSE}} = \sigma_s^2 \mathbf{D_F} \widetilde{\mathbf{H}}^H \left[\sigma_s^2 \mathbf{D_F} \widetilde{\mathbf{H}} \widetilde{\mathbf{H}}^H + \sigma_n^2 \mathbf{D_F}\right]^{-1}. \tag{6}$$

Bearing in mind that each of the individual channel matrices $\widetilde{\mathbf{H}}_{p,q}$ is a circulant matrix and circulant matrices can be diagonalized through the pre-multiplication of the FFT matrix $\mathbf{F}$ and the post-multiplication of the IFFT matrix $\mathbf{F}^{-1}$, we may perform a few algebraic manipulations to simplify equation (6) to give

$$\widehat{\mathbf{W}}_{\mathbf{MMSE}} = \sigma_s^2 \mathbf{\Lambda}^H \left[\sigma_s^2 \mathbf{\Lambda} \mathbf{\Lambda}^H + \sigma_n^2 \mathbf{I}_{nN}\right]^{-1} \tag{7}$$

where

$$\mathbf{\Lambda} = \mathbf{D_F} \widetilde{\mathbf{H}} \mathbf{D_F}^{-1} = \begin{pmatrix} \mathbf{F}\widetilde{\mathbf{H}}_{1,1}\mathbf{F}^{-1} & \mathbf{F}\widetilde{\mathbf{H}}_{1,2}\mathbf{F}^{-1} & \cdots & \mathbf{F}\widetilde{\mathbf{H}}_{1,n}\mathbf{F}^{-1} \\ \mathbf{F}\widetilde{\mathbf{H}}_{2,1}\mathbf{F}^{-1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{F}\widetilde{\mathbf{H}}_{n,1}\mathbf{F}^{-1} & \cdots & \cdots & \mathbf{F}\widetilde{\mathbf{H}}_{n,n}\mathbf{F}^{-1} \end{pmatrix} \tag{8}$$

and the term $\mathbf{F}\widetilde{\mathbf{H}}_{p,q}\mathbf{F}^{-1}$ is an $N$-by-$N$ diagonal matrix with the $N$ elements of the discrete frequency response of the channel between the $q$th transmit antenna and the $p$th receive antenna forming the diagonal. Therefore, the only information required to construct an MMSE equalizer in a MIMO SC-FDE system is the frequency response of the channel and the variance of the Gaussian noise process.

## 4 Results

In order to analyze the performance of the MMSE equalizer, Monte-Carlo performance simulations were constructed. The bit error rate (BER) and block error rate (BLER) were analyzed for a SISO SC-FDE system and two MIMO SC-FDE systems. Each system was simulated over a range of SNR values where 20,000 realizations of a five-tap Rayleigh-fading exponentially-decaying channel were simulated for each point to ensure an adequately large ensemble over which the BER and BLER of the system could be averaged. It was assumed that the receiver had perfect knowledge of the channel and the channel remained static for one transmitted block interval. Sixty-four BPSK information symbols were transmitted per block per antenna. The performance curves of the SC-FDE systems were compared to that of a SISO system employing a maximum-likelihood sequence estimation (MLSE) equalizer with perfect knowledge of the channel and all other relevant parameters equal to those of the SC-FDE systems, a good comparison since the MLSE equalizer gives optimal performance in terms of the BLER for a system with ISI. The results are illustrated in Figure 3. It is interesting to note that although the power of the transmitted signal per transmit antenna decreases as the numbers of transmit and receive antennas increase, the BER performance of the SC-FDE systems only differ from each other by less than one dB, a testament to the efficiency of the MMSE equalizer.

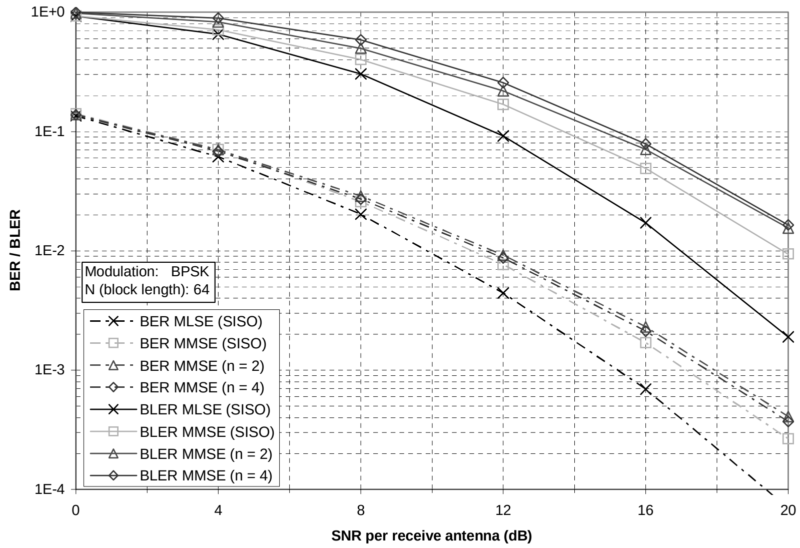


Figure 3: BER and BLER performance curves of several SC-FDE systems.

## 5 Conclusions

In this paper, a method was given to construct an MMSE equalizer for a MIMO SC-FDE system. Monte-Carlo simulations were employed to analyze the BER and BLER performances of several uncoded SC-FDE systems. The results of this analysis demonstrated a difference in performance from that of the MLSE solution of only a few dB.

There is much room for further research in the area of MIMO SC-FDE techniques and systems. Areas of future research may include SC-FDE architectures utilizing space-time codes to glean the maximum benefit from channel diversity as well as iterative SC-FDE encoding/decoding techniques.

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