Application of Nested Wigner Distributions to Radar Signal Analysis

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Abstract: This paper aims to outline some interesting properties of Nested Wigner Distributions. A number of features of such functions that can be useful in radar signal analysis are demonstrated with suitably simulated examples.

1. Introduction

The Nested Wigner Distributions (NWD) is a multi-dimensional extension to the conventional Wigner Distributions in the sense that the distributions are obtained by nesting a WD operation inside another WD operation (possibly more than one times) [1][2]. This results in the generation of a new type of distributions, which inter-relate many useful physical quantities: time, frequency, time delay, and frequency lag (Doppler). The order of the nest determines the dimension of the distribution, which increases with a factor of power of two with each order increment.

Many application areas can be identified where multi-dimensional distributions have gained much importance. WD has been applied to 2-D signals quite frequently in the past, in optics for example, where WD of a 2-D image results in a 4-D distribution [3] [4]. Following an analogy from this, it is well known that joint time-frequency analysis of radar signals generates a three-dimensional image cube that can be used for the removal of ambiguities related to complex-target maneuvers and structure [5] [6].

NWDs provide a unique method to inter-relate such dual (in the Fourier sense) variables. In the following sections we will look into ways that these inter-relations and associated properties may be applied for processing and analyzing radar signals.

2. Nested Wigner Distributions

For a time-domain signal *s(t)* whose Fourier transform is *S(ω)*, the WD is given by

$$W(t,\omega) = \int s\left(t+\frac{\tau}{2}\right) s^*\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau}\, d\tau. \tag{1}$$

(1) presents a joint time-frequency distribution, where τ is the time-lag. The Ambiguity Function (AF) (which is a joint lag-Doppler distribution) is given by:

$$A(\tau,\theta) = \int s\left(t+\frac{\tau}{2}\right) s^*\left(t-\frac{\tau}{2}\right) e^{jt\theta}\, dt, \tag{2}$$

where, θ is the 'Doppler'. The Wigner distribution and the Ambiguity Function form a two-dimensional Fourier transform pair. In the same way that *s(t)* and *S(ω)* are a Fourier pair, and we call their Wigner distribution a joint time-frequency distribution, it is natural to say that since Wigner distribution and Ambiguity function are 2-D Fourier pair, applying Wigner to itself will yield a joint Wigner-Ambiguity distribution. As will be seen shortly, this is a distribution defined in a four-dimensional space and inter-relates time, frequency, time-lag and Doppler. The mathematical expression for 1st order NWD is thus given as

$$W(t,\omega,\tau,\theta) = \iint W\left(t+\frac{\rho_1}{2},\omega+\frac{\rho_2}{2}\right) W^*\left(t-\frac{\rho_1}{2},\omega-\frac{\rho_2}{2}\right) e^{-j(\rho_1\theta+\rho_2\tau)} d\rho_1 d\rho_2 \tag{3}$$

where ρ_1 and ρ_2 are first-order (physical) time and (physical) frequency lags respectively (assuming the original τ, θ are zero-order). The same result can be obtained from a 2-D Ambiguity function:

$$W(t,\omega,\tau,\theta) = \iint A\left(\tau - \frac{\gamma_1}{2}, \theta - \frac{\gamma_2}{2}\right) A^*\left(\tau + \frac{\gamma_1}{2}, \theta + \frac{\gamma_2}{2}\right) e^{-j(\gamma_1\omega + \gamma_2 t)} d\lambda_1 d\lambda_2 \qquad \textbf{(4)}$$

Similar to the 2-D Wigner, NWD also shows some interesting properties, like symmetry, periodicity, inversions up to a constant and more noticeably 'marginals'. Here, we will show some cases of the marginal properties, which are subsequently used in our application examples. 2-D Wigner distributions satisfy two one-dimensional (time and frequency) marginals, but in the case of 4-D or higher dimensional Wigner distributions, one can obtain many multi-dimensional marginals along various running variables. For n^{th}-order NWD, the number of possible m-dimensional marginals is

$$N_m^n = \frac{2^{n+1}!}{m!(2^{n+1} - m)!} \qquad \textbf{(5)}$$

1st-order NWD satisfies four 3-D, six 2-D and four 1-D marginals. The 2-D marginals, one along time and frequency, and the other along lag and Doppler, give the squared-magnitude Wigner and Ambiguity functions respectively. The 3-D marginals are obtained by integrating the distributions along one chosen dimension. The following two are the most interesting ones for radar signal processing applications:

$$|G(\tau,\theta,t)| = \int W(t,\omega,\tau,\theta)\, d\omega \qquad \textbf{(6)}$$

$$|G(\omega,\tau,\theta)| = \int W(t,\omega,\tau,\theta)\, dt \qquad \textbf{(7)}$$

3. Joint Time-Frequency Analysis for Radar Signals

Joint Time-Frequency (JTF) analysis has been very efficiently used by the radar signal processing community for some time now. In this section we discuss some interesting properties of higher-order NWDs, their marginals and their potential for radar signal processing. Radar transmits a series of pulse signals, and then gathers and analyses received data (reflected and/or noise) to retrieve as much information about potential target(s) as possible. This function is based on the fact that the time-delay between the transmitted and reflected signals is related to range of the target, where the range is the distance from radar to target measured along the radar line of sight (LoS); and the increase (or decrease) in the frequency of the transmitted signal, called Doppler shift, gives information about target motion and maneuvering in the cross-range dimension. This is the dimension transverse to the radar LoS. The radar range profile is the distribution of target reflectivity along the radar LoS and the radar Doppler profile is the distribution along Doppler shifts. Finally combining these two profiles, the lag-Doppler plane $H(\tau,\theta)$ (called the radar function) forms the radar image.

For practical targets, the scattering characteristics are highly complex. The resonance in hollow structures (eg: engine inlets) and material or structural dispersion as a function of frequency may give rise to extended (more widely spread) returns rather than sharp and impulsive in the range domain [7]. Similarly, the Doppler profile may become blurred when the target exhibits complex maneuvers. Highly sophisticated signal processing techniques have been used to limit these difficulties with little improvement in image formation [5], but application of JTF processing has improved the situation considerably.

JTF processing is applied to the range (time-lag) axis of the conventional range/cross-range (lag-Doppler) image to gain an extra frequency dimension forming a 3-D cube.

$$H(\tau,\theta) \rightarrow G(\tau,\omega,\theta) \qquad \textbf{(8)}$$

This extra frequency axis helps in identifying scattering effects mentioned above. Similarly, an application of JTF processing on the Doppler domain of the conventional 2-D image results in an extra time dimension, which helps in observing time-varying behavior of the moving targets.

$$H(\tau,\theta) \rightarrow G(\tau,\theta,t) \qquad \textbf{(9)}$$

It should be interesting to note that the first-order NWD is a joint function of time, frequency, lag and Doppler and it can easily be computed from the lag-Doppler plane as shown in (4). The two of its 3-D marginals in equations (6) and (7) are exactly the same matrices that are used for these two purposes. Conventionally, scattering and target motion compensations are performed as separate processing stages (perhaps by separate modules) and the results are merged after the processing is completed. First-order NWD provides a unique method to achieve time-frequency distributions for both types of compensation in one combined computation. Also, this can help relate the effects from scattering and motion, which may lead to better prediction of target maneuvers. Figure 1 shows 3-D marginals obtained from first-order NWD of sinusoidally modulated and linear chirps.

In the examples shown below (Figure 1), relationships between three important quantities (time, lag and Doppler) are shown graphically that can be achieved using the 3-D marginals obtained form 1st-order NWDs. In the first row of figures, a 3-D relationship surface is plotted for a sinusoidally modulated chirp. The relationship variation between these quantities in terms of signal energy concentration, spread and density is obvious. Figure 1(c) shows a 2-D slice taken from the 3-D volume at the location shown in Figure 1(b) that is parallel to the time-time-lag dimension. The slice shows that at this particular Doppler shift, the energy of the distribution is concentrated on the lower time-lag values and middle time-values. The energy distribution is not concentrated very well, rather spread forming a circular density function. The energy distribution at various other locations can easily be guessed by imaging a cross-section in the above mentioned way.

Inter-relations between quantities can be exploited more by taking more than one 2-D slices that are perpendicular to each other (Figure 2). This approach helps in the integration of information obtained from individual 2-D planes. Time (or frequency) varying behavior of energy distributions can be obtained easily by moving the 2-D plane intersection location as required.

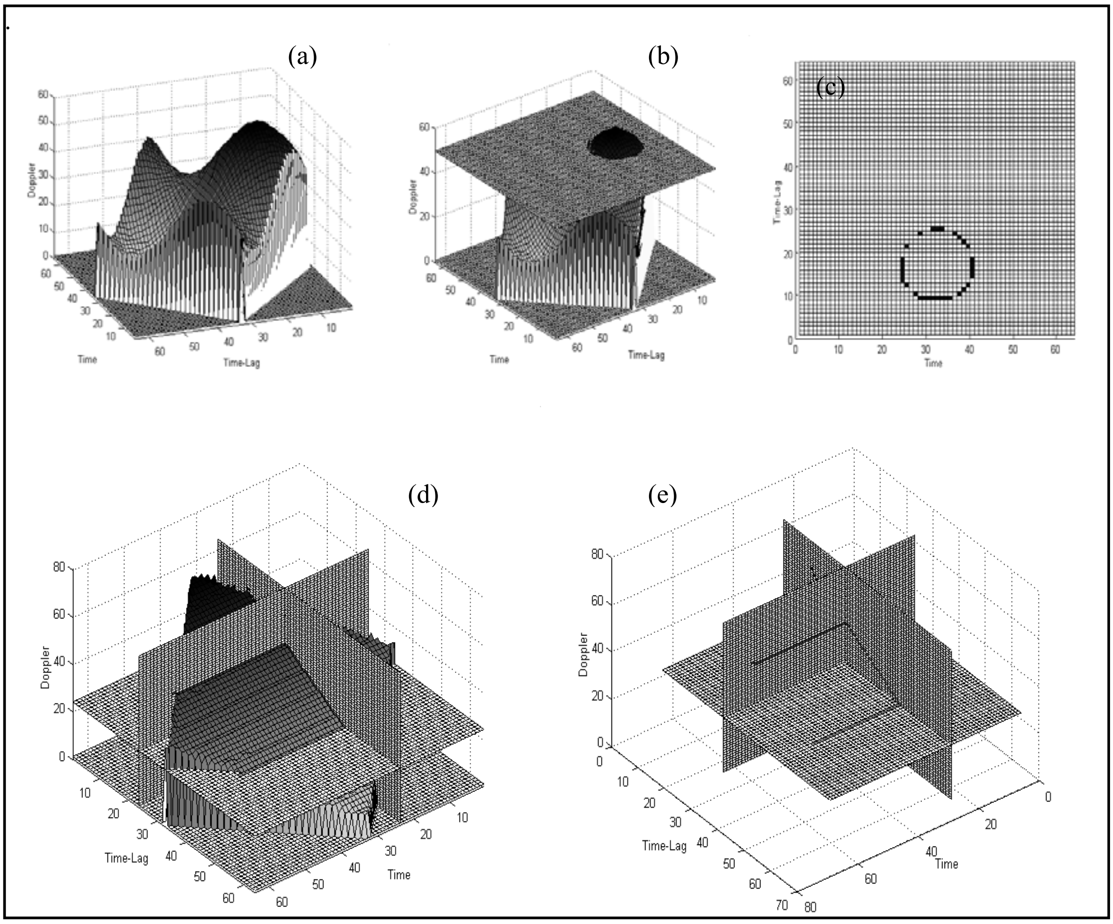

Figure 1: 3-D Time-Lag-Doppler marginals obtained from 4-D space; (a,b,c) Sinusoidally modulated chirp; (c,d) Linear chirps

First-order NWD can be exploited for obtaining *"localized variance"*. This statistical relationship gives localized inter-dependence of two physical quantities. This is useful in the sense that it gives time and/or spectral varying nature of covariance between two or more quantities. When applied to 4-

D NWD results in a 2-D covariance distribution. There can be various combinations, but the one given below gives the localized temporal-spectral covariance of lag and Doppler.

$$Cov_{\tau\theta|t,\omega} = \langle \tau\,\theta \rangle_{t,\omega} - \langle \tau \rangle_{t,\omega} \langle \theta \rangle_{t,\omega} \quad \textbf{(10)}$$

where,

$$\langle \tau \rangle_{t,\omega} = \frac{1}{|W(t,\omega)|^2} \iint \tau\, W(t,\omega,\tau,\theta)\, d\theta\, d\tau \quad \textbf{(11)}$$

$$\langle \theta \rangle_{t,\omega} = \frac{1}{|W(t,\omega)|^2} \iint \theta\, W(t,\omega,\tau,\theta)\, d\theta\, d\tau \quad \textbf{(12)}$$

The localized spread of lag-Doppler plane is easily obtained through the conditional standard variations of τ and θ as follows. Figure 2(a) shows the spreads obtained by the above-listed operations.

$$\sigma^2_{\tau|t,\omega} = \left\langle \tau^2 \right\rangle_{t,\omega} - \left\langle \tau \right\rangle^2_{t,\omega} \quad \textbf{(13)} \qquad \sigma^2_{\theta|t,\omega} = \left\langle \theta^2 \right\rangle_{t,\omega} - \left\langle \theta \right\rangle^2_{t,\omega} \quad \textbf{(14)}$$

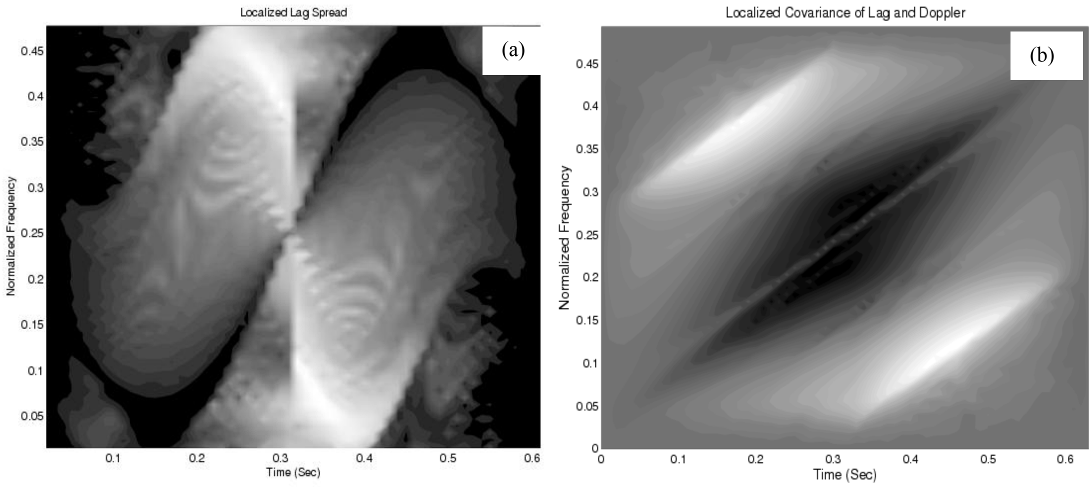

Figure 2: Localized Time-lag spread for (a) sinusoidally modulated chirp and (b) a linear chirp.

4. Conclusions

We discussed in this paper Nested Wigner Distributions that are a new multi-dimensional distribution and offers very interesting inter-relations as well as properties. We presented these properties in the context of application to radar signal processing. Also, it is shown through the example of a simple nature that marginals in different dimensions and directions offer greater flexibility in target scattering and complex motion compensations.

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